Bottom-up rewriting is inverse recognizability preserving

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PLAN

1.1-RECALLS-SEMI-THUE SYSTEMS

A rewrite rule over the alphabet A is a pair

$$\ell \to r$$

of words in A^*.

A semi-Thue system is a pair (S, A) where S a set of rewrite rules built upon the alphabet A.

For every $f, g \in A^*$,

$$f \to_S g$$

iff there exists $\ell \to r \in S$ and $\alpha, \beta \in A^*$ such that

$$f = \alpha\ell\beta \;\&\; g = \alpha r\beta.$$

We call $\to_S^*$ the derivation generated by S.

1.2-RECALLS- TERM-REWRITING SYSTEMS

A rewrite rule is a pair

$$\ell \rightarrow r$$

of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ which satisfy $\mathcal{V}\text{ar}(r) \subseteq \mathcal{V}\text{ar}(\ell)$.

A term rewriting system is a pair $(\mathcal{R},\mathcal{F})$ where $\mathcal{F}$ is a signature and $\mathcal{R}$ a set of rewrite rules over the signature $\mathcal{F}$. For every $s, t \in \mathcal{T}(\mathcal{F},\mathcal{V})$,

$$s \rightarrow_{\mathcal{R}} t$$

iff there exists $\ell \rightarrow r \in \mathcal{R}$ a context $C[]$ and a substitution σ such that

$$s = C[\ell\sigma] \ \& \ t = C[r\sigma].$$

We call $\rightarrow^*_{\mathcal{R}}$ the derivation generated by $\mathcal{R}$.

2.1-INTRODUCTION- PROBLEMS

Given a system $\mathcal{R}$ and a set of terms T, we define

$$(\rightarrow^*_{\mathcal{R}})[T] = \{s \in \mathcal{T}(\mathcal{F}) \mid s \rightarrow^*_{\mathcal{R}} t \text{ for some } t \in T\}$$

Problem: under which hypothesis over $\mathcal{R}$ does it hold that, for every recognizable set T, $(\rightarrow^*_{\mathcal{R}})[T]$ is recognizable too.

2.2-INTRODUCTION- MOTIVATIONS

-rational subsets of a monoid $M = A^*/\leftrightarrow^*_S$
application to resolution of equations with rational constraints in M.

-decidability of the accessibility problem for $\rightarrow_{\mathcal{R}}$

-decidability of the common-ancestor problem for $\rightarrow_{\mathcal{R}}$

-sequentiality problems:

- computation of a needed redex in a term t w.r.t. $\mathcal{R}$
- decidability of the sequentiality property for $\mathcal{R}$

-decidability of the termination problem.

2.3-INTRODUCTION- KNOWN RESULTS

Two kinds of results.

First kind: syntactical condition over $\mathcal{R}$

Generic theorem:

if $\mathcal{R}$ has property P, then, for every recognizable set T,

$(\rightarrow^*_{\mathcal{R}})[T]$ is recognizable too.

2.3-INTRODUCTION- KNOWN RESULTS

First kind references (syntax):

- cancellation rules Benois-Sakarovitch, IPL 86
- basic semi-Thue systems Benois, RTA 87
- left-basic semi-Thue systems Sakarovitch, PHD, 79
- ground term-rewriting systems

Dauchet-Heuillard-Lescanne-Tison, Inf. and Comput. 87

- linear shallow term-rewriting systems Comon, LICS 95
- linear growing term-rewriting systems Jacquemard, RTA 96
- left-linear growing term-rewriting systems

Nagaya-Toyama,Inf. and Comput. 02

- left-linear inverse finite-path overlapping TRS

Takai-Kaji-Seki,Sci. Math. Jap., 2006

2.3-INTRODUCTION- KNOWN RESULTS

Second kind: use a special strategy in derivations

Generic theorem (for the strategy $\mathcal{S}$):

For every TRS $\mathcal{R}$ and every recognizable set T, $({}_{\mathcal{S}}\!\rightarrow^*_{\mathcal{R}})[T]$ is recognizable too.

2.3-INTRODUCTION- KNOWN RESULTS

Second kind references (strategy):

- one-pass term rewriting

Fulop-Jurvanen-Steinby-Vagvolgyi, MFCS 98

- concurrent term rewriting

Seynhaeve-Tison-Tommasi, FCT 99

- “bottom-up derivations”

Rety-Vuotto, JSC 05.

2.4-INTRODUCTION- NEW RESULTS

We define a new notion of k-bottom-up derivation, denoted by ${}_k\!\!\longrightarrow^*_{\mathcal{R}}$.

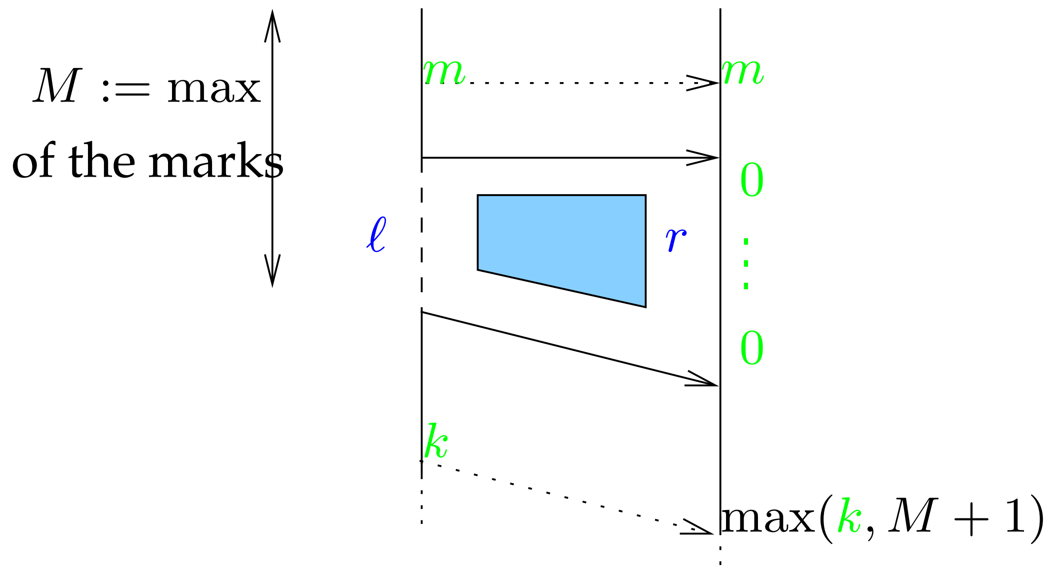
Theorem 1 *Let $\mathcal{R}$ be some linear rewriting system over the signature $\mathcal{F}$, let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k \geq 0$. Then, the set $({}_k\!\!\longrightarrow^*_{\mathcal{R}})[T]$ is recognizable too.*

We then introduce the class of Bottom-Up systems as the class of all systems for which the above strategy is complete.

2.5-INTRODUCTION- METHODS

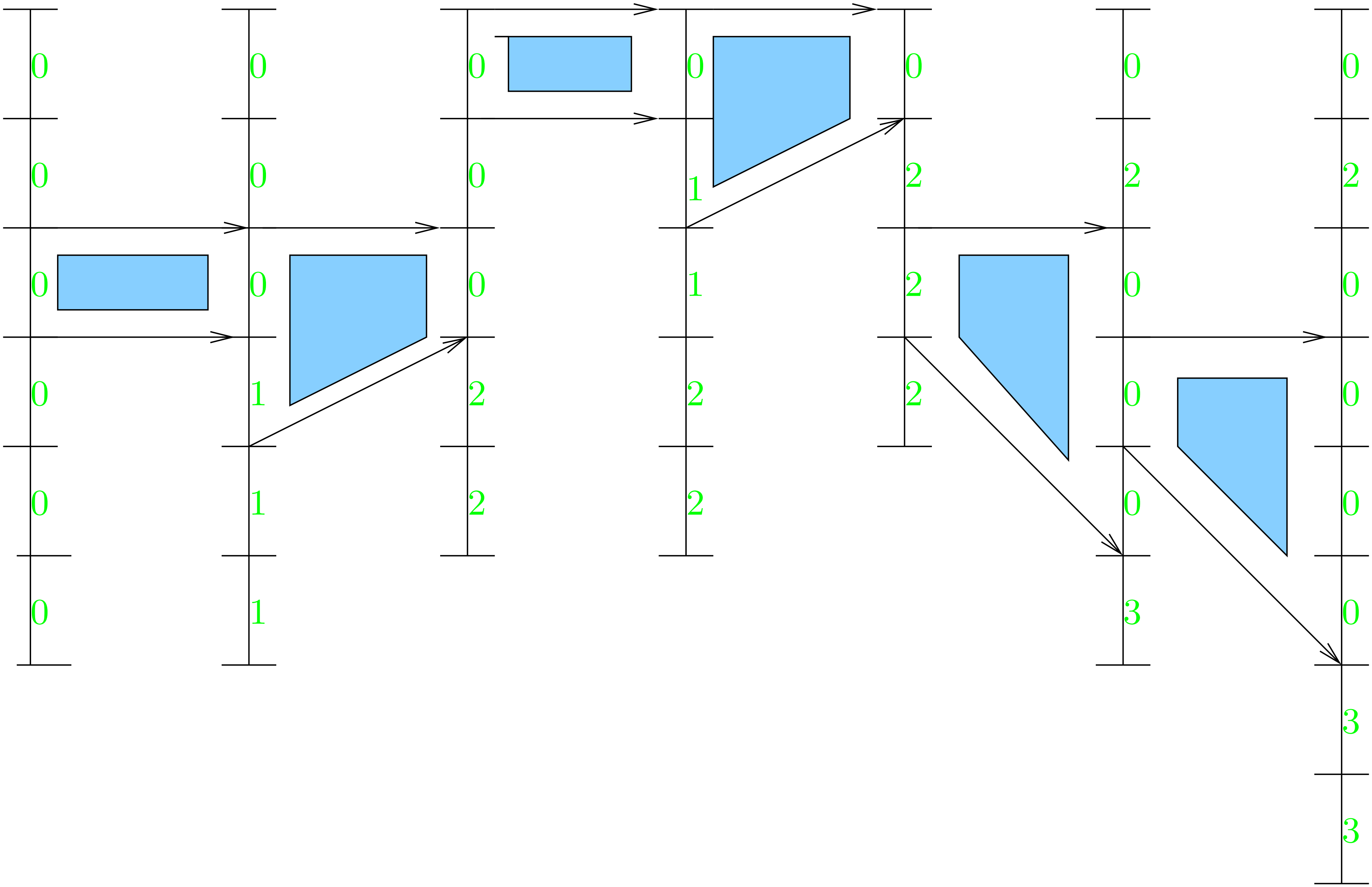
1- Define a marking operation: the marks are integers that measure the amount of top-down (i.e. “forbidden”) sequence of rules.
Bottom-up derivation are those derivations with small marks.
2- Reduce the preservation property to the same property for ground sytems: a bottom-up derivation can be simulated by a ground derivation.

3.1-Marked DERIVATIONS- Unary terms



3.1-MARKED DERIVATIONS- UNARY TERMS

A derivation graph.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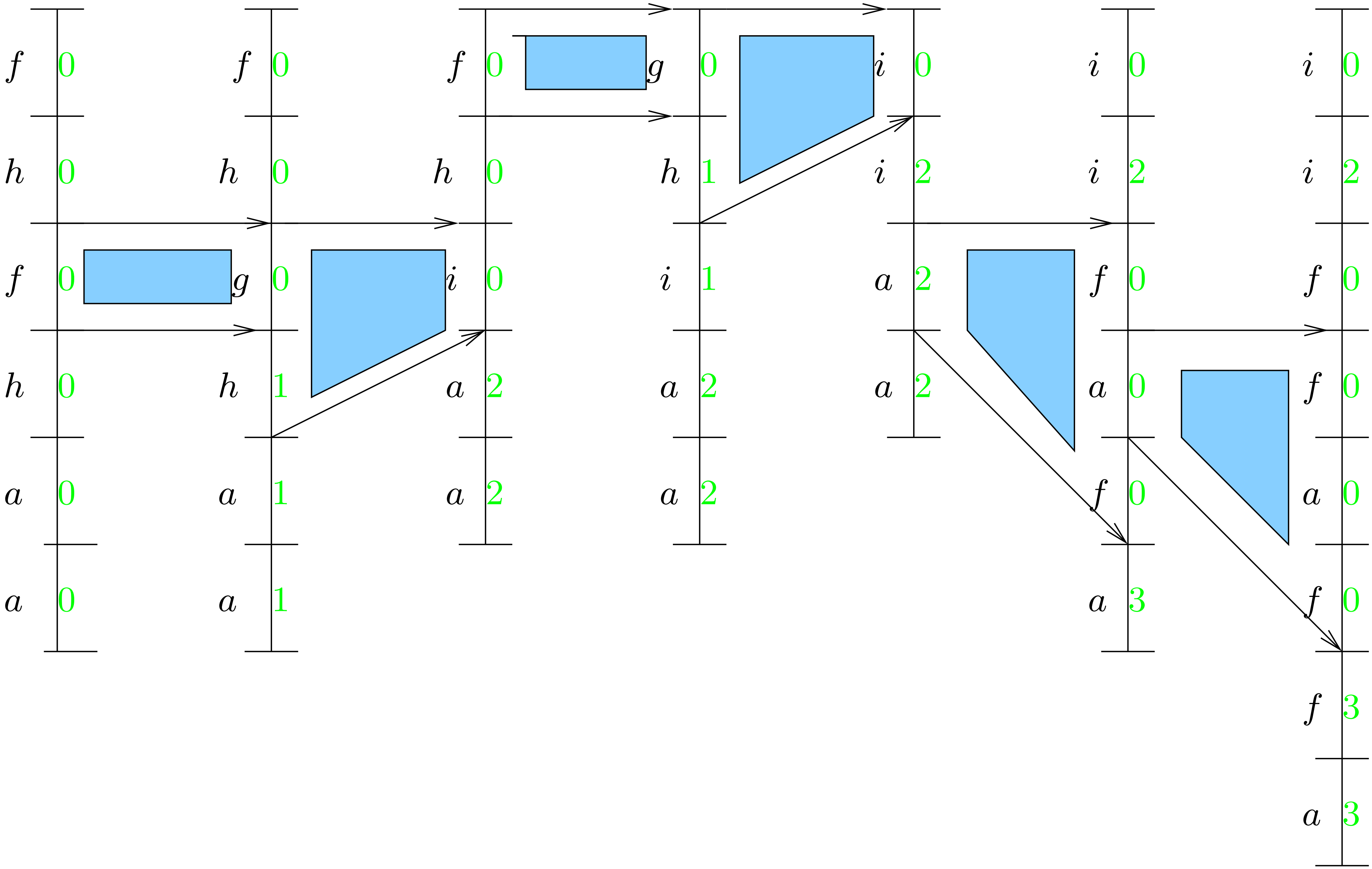
3.1-Marked DERIVATIONS- Unary terms

A derivation.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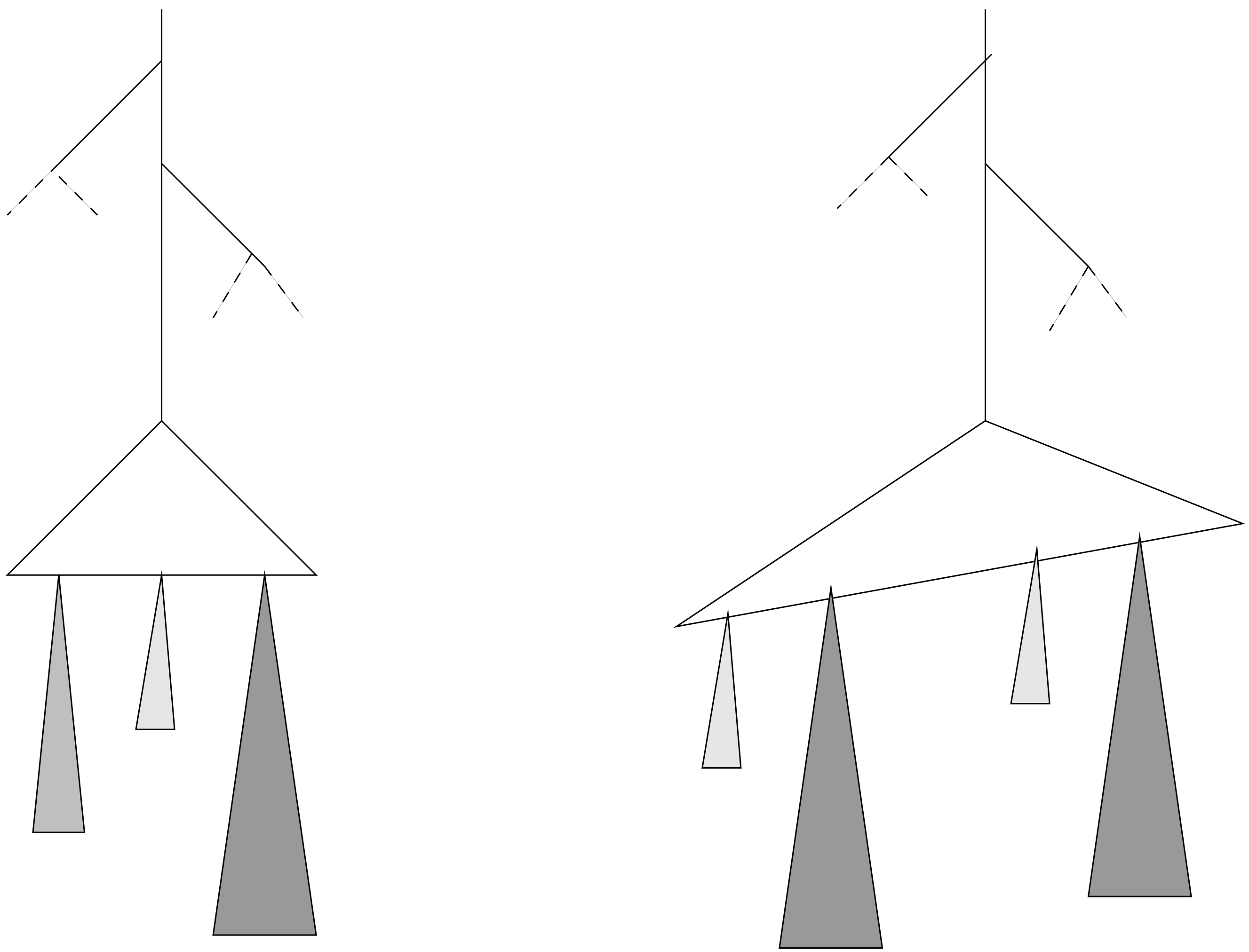
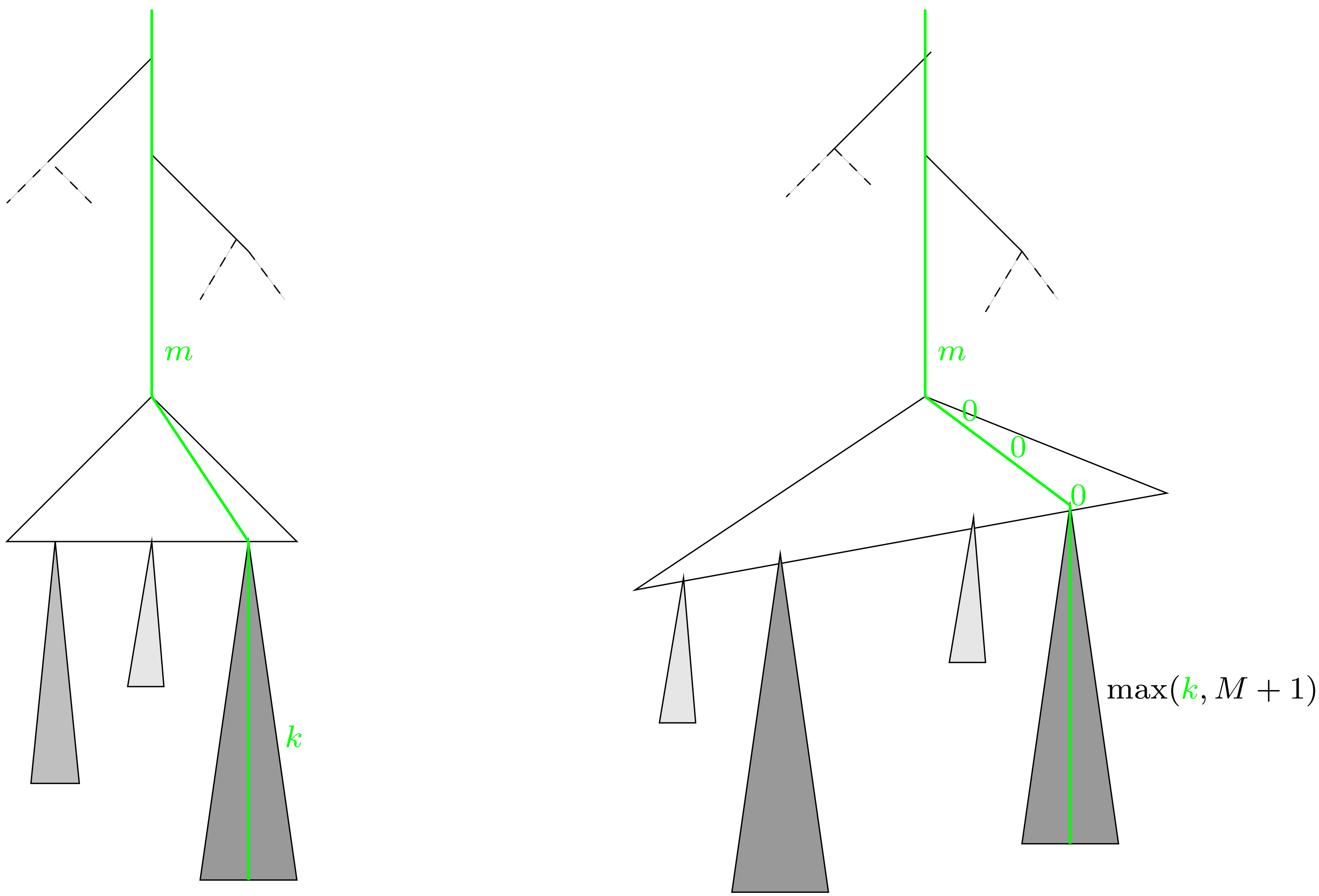
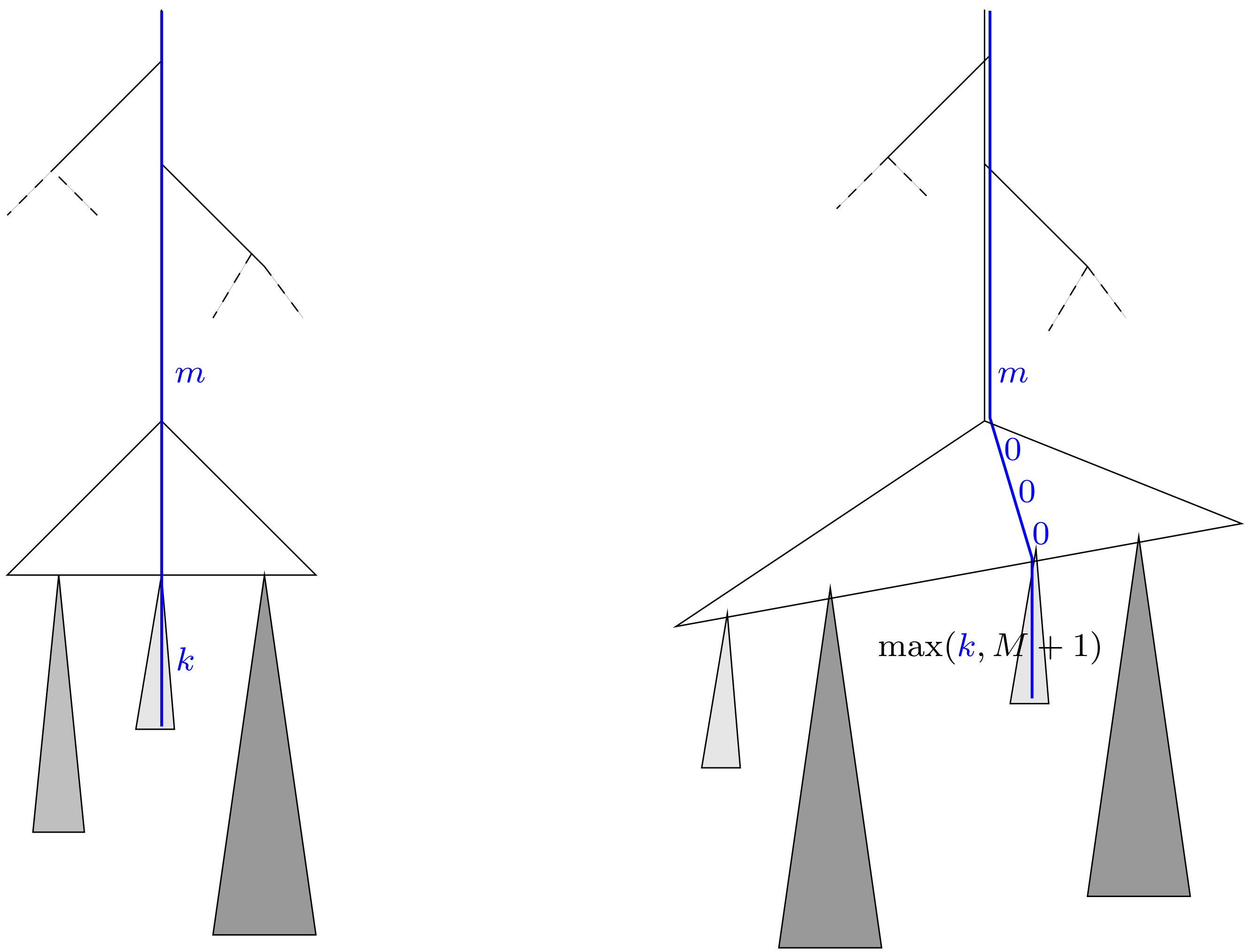
3.2-Marked DERIVATIONS- General terms



3.2-Marked DERIVATIONS- General terms



3.2-Marked DERIVATIONS- General terms



4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

Definition 2 *A derivation $s \to_{\mathcal{R}}^{*} t$ is weakly bottom-up iff, in the corresponding marked derivation, for every application of rule, the minimum mark of the lhs is 0.*

Let $k \geq 0$.

Definition 3 *A derivation $s \to_{\mathcal{R}}^{*} t$ is k bottom-up iff, in the corresponding marked derivation , for every application of rule, the minimum mark of the lhs is 0 and the maximum mark of the term is $\leq k$.*

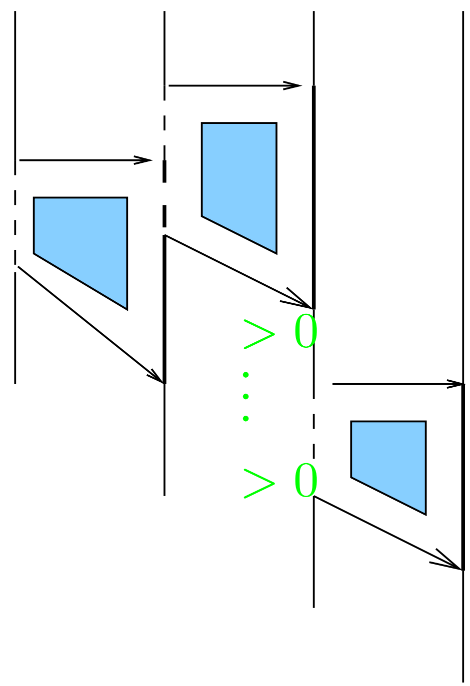
Notation:

$$s \ {}_{k}{\to}_{\mathcal{R}}^{*} \ t$$

means that there exists a k bottom-up derivation from s to t.

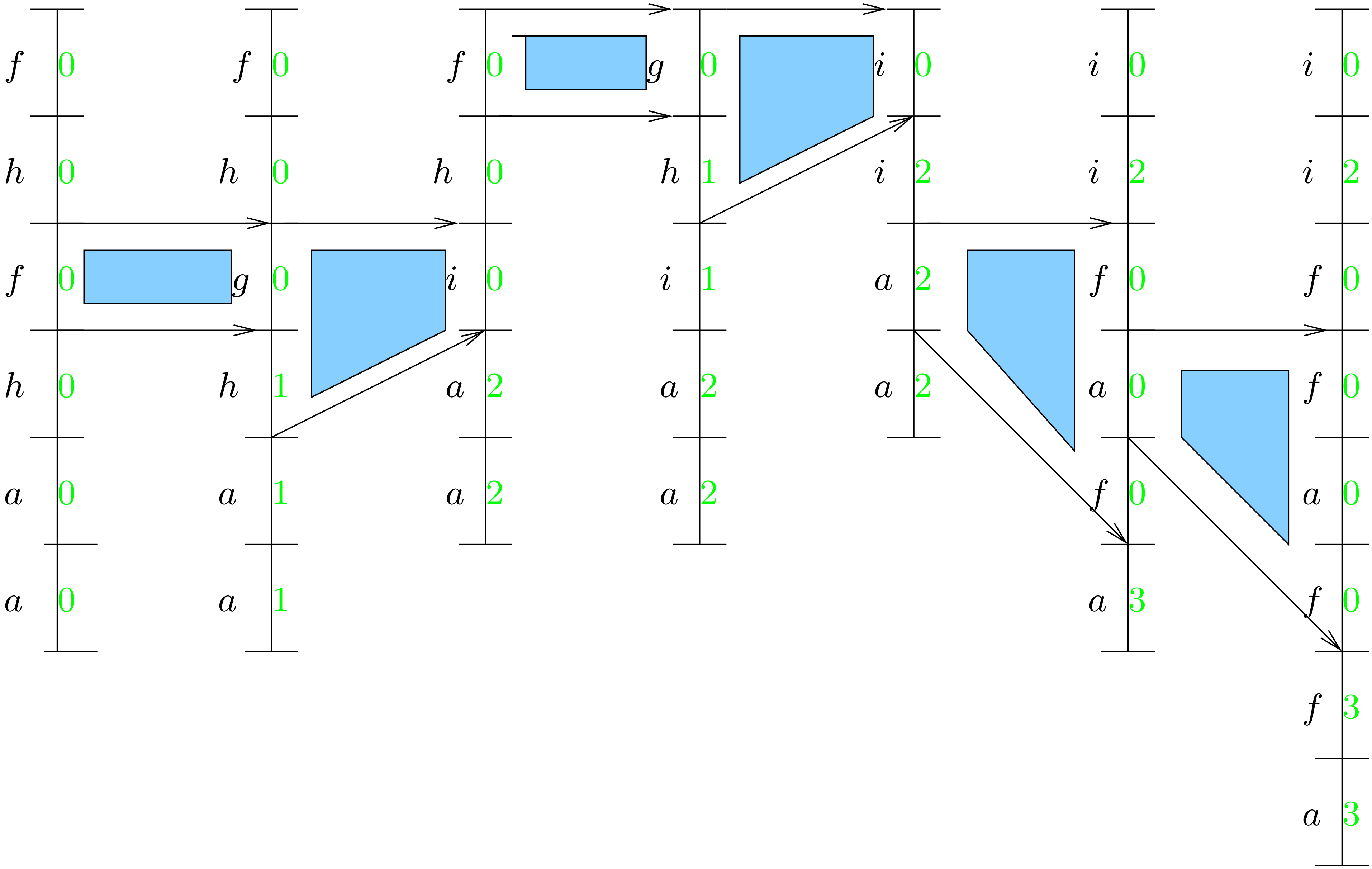
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A generic non weakly bottom-up derivation.



4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete non weakly bottom-up derivation.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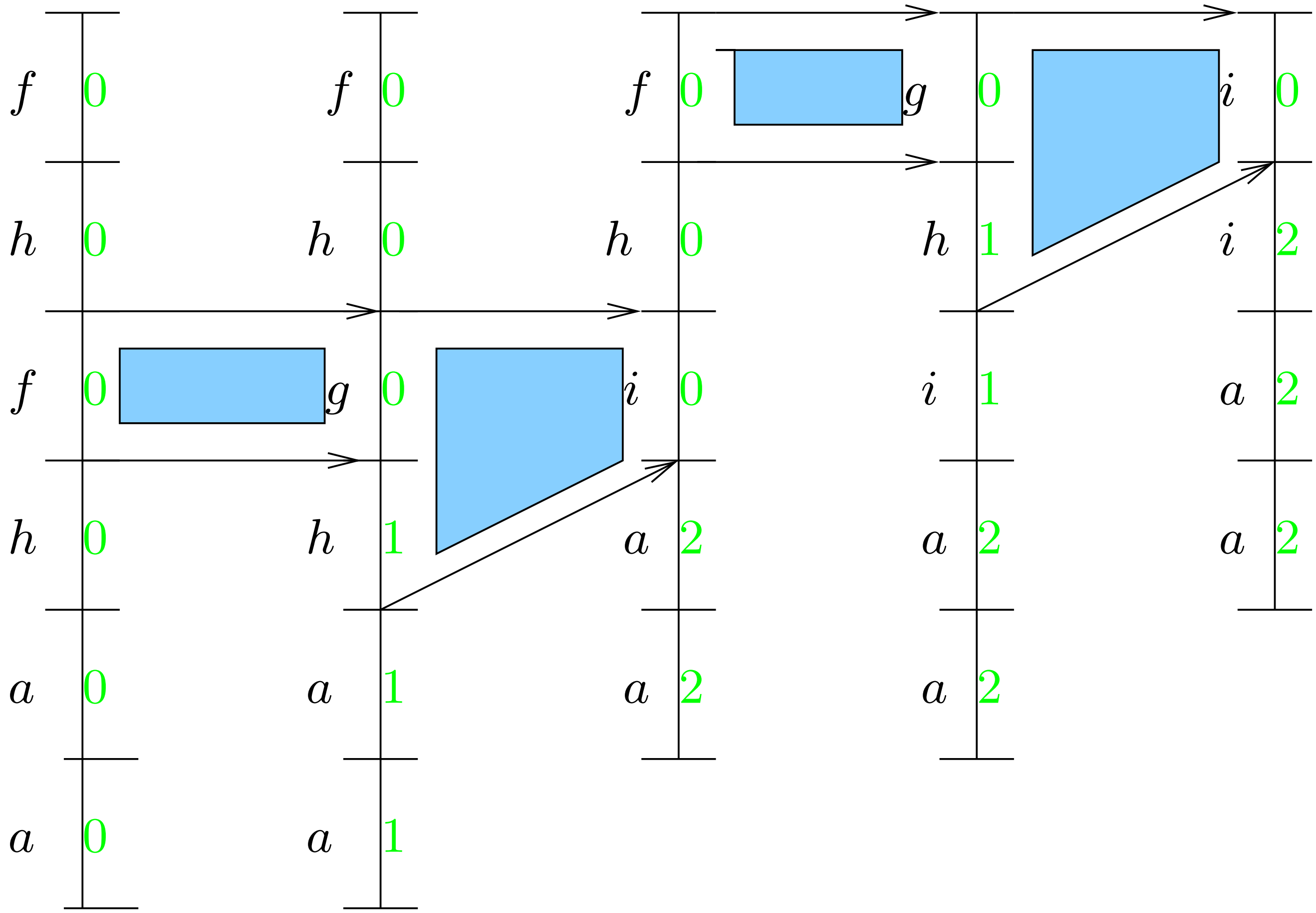
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete derivation:

- it is weakly bottom-up,
- it is not BU(1),
- it is BU(2),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4.2 -BOTTOM-UP DERIVATIONS- BOTTOM-UP SYSTEMS

Definition 4 *Let $k \geq 0$. A system $(\mathcal{R}, \mathcal{F})$ is called k-Bottom-Up iff, it is linear and for every $s, t \in \mathcal{T}(\mathcal{F})$*

$$s \rightarrow^*_{\mathcal{R}} t \Rightarrow s \;_k\!\rightarrow^*_{\mathcal{R}} t$$

i.e. if $\mathcal{R}$ is linear and the k-BU strategy is complete for $\mathcal{R}$.

4.3 -BOTTOM-UP DERIVATIONS- KNOWN SUBCLASSES

The following classes of systems are bottom-up:

- every left-basic semi-Thue system is BU(1)
- every linear growing Term Rewriting system is BU(1)
- every linear FPO^{-1} Term Rewriting system is in $\bigcup_{k\geq 0}$ BU(k).

5.1-PRESERVATION OF RATIONALITY-THE RESULT

Theorem 1
Let $\mathcal{R}$ be some linear rewriting system over the signature $\mathcal{F}$, let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k \geq 0$. Then, the set $({}_k{\rightarrow}^*_{\mathcal{R}})[T]$ is recognizable too.

Theorem 5 *Let $k \geq 0$, let $\mathcal{R}$ be some* BU(k) *rewriting system over the signature $\mathcal{F}$ and let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$. Then, the set $(\rightarrow^*_{\mathcal{R}})[T]$ is recognizable too.*

5.2.2-PRESERVATION OF RATIONALITY-CONSTRUCTION OF $\mathcal{S}$

General idea:

- Simulate a bottom-up derivation, by a derivation where the substitutions used have a bounded depth
- The deeper part of the substitution is replaced by a state of the finite automaton recognizing T.

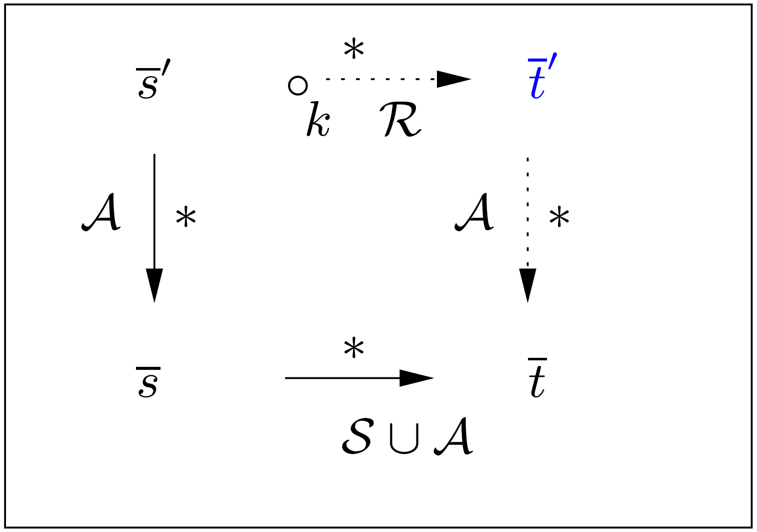
5.2.2-PRESERVATION OF RATIONALITY-CONSTRUCTION OF $\mathcal{S}$

Let d $:= \max\{dpt(\ell) \mid \ell \to r \in \mathcal{R}\}$.
The system $\mathcal{S}$: consists of all the rules

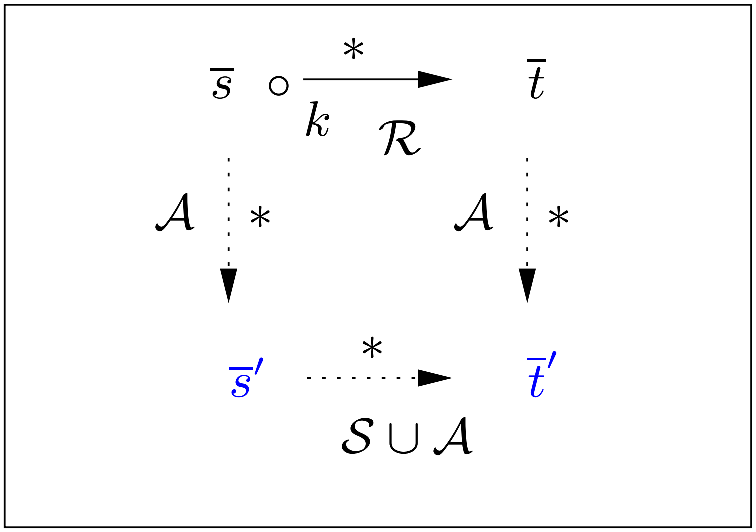
$$\overline{\ell}\overline{\tau} \to r\overline{\overline{\tau}}$$

where $\ell \to r$ is a rule of $\mathcal{R}$, $\overline{\tau}, \overline{\overline{\tau}}$ are marked substitutions, with marks $\leq k$, $dpt(\overline{\tau}) \leq k \cdot$ d and $\overline{\ell}\overline{\tau} \to r\overline{\overline{\tau}}$ is a one-step, $k-$ bu, marked derivation.

5.2.3-PRESERVATION OF RATIONALITY- SIMULATION LEMMAS



Lifting $\mathcal{S} \cup \mathcal{A}$



Projecting $\mathcal{R}$

5.2.4-PRESERVATION OF RATIONALITY- CONCLUSION

Since $\mathcal{A}$ and $\mathcal{S}$ are *ground* rewriting-systems, it is known that $\rightarrow^*_{\mathcal{S}\cup\mathcal{A}}$ inverse-preserves recognizability. By the simulation lemmas:

$$({}_k\!\rightarrow^*_{\mathcal{R}})[T] = (\rightarrow^*_{\mathcal{S}\cup\mathcal{A}})[Q_f^{\leq k}] \cap \mathcal{T}(\mathcal{F})$$

Hence $({}_k\!\rightarrow^*_{\mathcal{R}})[T]$ is recognizable.

6.1-COMPLEXITY/DECIDABILITY- BOTTOM-UP

Theorem 6 *The BU(1) property is undecidable for semi-Thue systems.*

6.2-COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

Definition 7 *Let $k \geq 0$. A system $(\mathcal{R}, \mathcal{F})$ is called strongly k-Bottom-Up iff, it is linear and for every weakly bottom-up marked derivation*

$$s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} s_i \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} s_n = t$$

if s_0 has only null marks, then all the s_i have all their marks $\leq k$.

Theorem 8 *The SBU(k) property is decidable for Term Rewriting systems.*

Follows easily from theorem 1.

4.3 -COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

- every left-basic semi-Thue system is SBU(1)
- every linear growing Term Rewriting system is SBU(1)
- every linear FPO^{-1} Term Rewriting system is in $\bigcup_{k\geq 0}\mathrm{S}BU(k)$.

7- PERSPECTIVES

- Extend the notion and the results to left-linear, non right-linear systems
- Study a dual notion of top-down derivations