Bordeaux University
Master in Computer Science, S1, 2014/2015

LOGICS, J1IN7M21

Examination on 15/12/2014, 8h 30-11h 30

Subject from M. Sénizergues ; all documents allowed ;
Recommended duration for this part : 1h 30.
The mark for this half of the exam will be $\min\{exo1 + exo2 + exo3 + exo4, 10\}$.

Exercise 1 (4 pts)
In the following A, B, C are propositional variables and P, Q are unary predicate symbols.
1- For the following sequents, either give a proof within LK or show that such a proof does not exist :

$$(\neg A) \rightarrow (B \vee C) \vdash A \vee ((\neg B) \rightarrow C)$$

$$\vdash (\forall x((P(x) \rightarrow Q(x)) \rightarrow P(x))) \rightarrow \forall y P(y)$$

2- Give a proof within LJ for the following sequent :

$$\vdash ((A \rightarrow B) \rightarrow A)) \rightarrow \neg\neg A$$

Exercise 2 (4 pts)
In the following A, B are propositional variables.
1- Show, by *semantical* arguments, that

$$A \rightarrow B, (\neg A) \rightarrow B, \neg B \models \bot$$

2- Can you infer from question 1 above, that there exists a proof, within LJ, of the sequent

$$A \rightarrow B, (\neg A) \rightarrow B, \neg B \vdash \bot ?$$

3- Give a proof within LJ, of the sequent

$$A \rightarrow B, (\neg A) \rightarrow B, \neg B \vdash \bot$$

Exercise 3 (4 pts)
In the following A is a propositional variable.
1- Show that the sequent $\vdash (\neg\neg A) \rightarrow A$ is not provable within LJ.
2- Let us examine, in the system LJ, the *left*-introduction (scheme of) rule for the negation connector. Is this scheme of rule *reversible* ?
3- Let us examine, in the system LJ, the *right*-introduction (scheme of) rule for the negation connector. Is this scheme of rule *reversible* ?

Exercise 4 (6 pts)
We give here a semantics-based algorithm that decides whether some special formulas are provable within LK. Let

$$\mathcal{S} := \langle R_1, R_2, \ldots, R_n; c_1, c_2, \ldots, c_m \rangle$$

be some signature with the arities

$$\langle r_1, r_2, \ldots, r_n; a_1, a_2, \ldots, a_m \rangle$$

and $a_1 = a_2 = \ldots = a_m = 0$ i.e. the c_j's are *constant* symbols.
Let $\psi(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ be some first-order formula over $\mathcal{S}$, without any quantifier, with free variables $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ and let

$$\Phi := \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_\ell \;\; \psi(x_1 \ldots x_k, y_1 \ldots, y_\ell). \tag{1}$$

1- Show that, if $\mathcal{A} := \langle A; R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, \ldots, R_n^{\mathcal{A}}; c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, \ldots, c_m^{\mathcal{A}} \rangle$ is a structure fulfilling

$$\mathcal{A} \models \Phi$$

then there exists a structure $\mathcal{A}'$ over $\mathcal{S}$, whose domain A' has cardinality less than $k + m$ and also fulfilling $\mathcal{A}' \models \Phi$.
Hint : consider a valuation ν that makes true the formula $\forall y_1 \ldots \forall y_\ell \; \psi(x_1 \ldots x_k, y_1 \ldots, y_\ell)$ and build a structure with domain $\{c_1^{\mathcal{A}}, \ldots, c_m^{\mathcal{A}}\}$ union the set $\{\nu(x_i) \mid 1 \leq i \leq k\}$.
2- Show that it is decidable whether a formula of the form (1) has a (classical) model.
3- Let us consider a formula of the form

$$\theta := \forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_\ell \;\; \psi(x_1 \ldots x_k, y_1 \ldots, y_\ell). \tag{2}$$

3.1 Explain how one can test whether such a formula is true in every structure over the signature $\mathcal{S}$.
3,2 Give an algorithm that takes in input a formula of the form (2) and outputs 1 (resp 0) iff the sequent $\vdash \theta$ is provable (resp. not provable) within LK.
3.3 Give a time (or space) complexity upper-bound for the algorithm you propose.