LOGICS

TD 4: Kripke structures

Exercice 4.1

Prove, in a semantical way , that $\Vdash \forall x\neg\neg(R(x) \vee \neg R(x))$

Exercice 4.2

1- Find some Kripke structure $\mathcal{K}$ which is a counter-model for

$$(\neg A \to B) \to (\neg B \to A)$$

2- Is the sequent $\vdash (\neg A \to B) \to (\neg B \to A)$ derivable within LK? within LJ?

Exercice 4.3 LJ versus LK

Let us consider the following sequents, that were seen to be derivable within LK (see exercises over chapter 2). For each of them, determine whether it is derivable within LJ.

1- $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
2- $\vdash ((P \to Q) \to P) \to P$
3- $\vdash \neg\forall x\neg R(x) \to \exists x R(x)$
4- $\vdash \exists x\forall y(R(y) \to R(x))$
5- $\forall x\neg R(x) \vdash \neg\exists x R(x)$
6- $\vdash \exists x(R(a) \vee R(b) \to R(x))$.

Exercice 4.4 Bisimulations

Let $\mathcal{K}_1, \mathcal{K}_2$ be Kripke structures over some propositional signature $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$. Let us use the notation $\mathcal{K}_i := (K_i, \leq_i, \Vdash_i)$ pour $i \in \{1, 2\}$. A binary relation $R \subseteq K_1 \times K_2$ is called a *bisimulation* iff it fulfills the three following properties B1,B2,B3:

(B1) $\forall(k_1, k_2) \in R, \forall Q \in \mathcal{Q}, k_1 \Vdash Q \Leftrightarrow k_2 \Vdash Q$

(B2) $\forall(k_1, k_2) \in R, \forall k_2' \in K_2$ such that $k_2 \leq_2 k_2'$,

$$\exists k_1' \in K_1 \text{ such that } (k_1', k_2') \in R \;\&\; k_1 \leq_1 k_1'.$$

(B3) $\forall(k_1, k_2) \in R, \forall k_1' \in K_1$ such that $k_1 \leq_1 k_1'$

$$\exists k_2' \in K_2 \text{ such that } (k_1', k_2') \in R \;\&\; k_2 \leq_2 k_2'.$$

1- Show that, if R is a bisimulation then, for every $(k_1, k_2) \in R$ and for every formula A,

$$k_1 \Vdash_1 A \Leftrightarrow k_2 \Vdash_2 A.$$

2- Show that, every Kripke structure that posesses a smallest element is bisimilar to a tree.

Exercice 4.5 A complete structure

Let $\mathcal{K} = (K, \leq, \mid\!\vdash\!-\,)$ be a Kripke structure over the propositional signature $\mathcal{Q}$ and let A be some formula over $\mathcal{Q}$. Let us denote by $SF(A)$ the set of all sub-formulas of A. For every $k \in K$ we note $S(k) := \{B \in SF(A) \mid k \mid\!\vdash\!- B\}$. Let us define a Kripke structure $\mathcal{K}^* = (K^*, \leq^*, \mid\!\vdash\!-^*)$ by:

$$K^* := \{S(k) \mid k \in K\}, \;\; S(k) \leq^* S(k') \Leftrightarrow S(k) \subseteq S(k')$$

and for every $Q \in \mathcal{Q}$

$$S(k) \mid\!\vdash\!-^* Q \Leftrightarrow Q \in S(k).$$

1- Show that, for every $B \in SF(A)$ and every $k \in K$,

$$k \mid\!\vdash\!- B \Leftrightarrow S(k) \mid\!\vdash\!-^* B$$

2- Deduce from the above question a semantical method for testing that, for every propositional formula A, whether $\mid\!\vdash\!- A$ holds (or not).

3- Show that, for every propositional formula A , $\vdash_{\mathrm{LJ}} A$ iff every finite Kripke structure $\mathcal{K}$ satisfies A.

4- Using exercise 4.4, show that, for every propositional formula A , $\vdash_{\mathrm{LJ}} A$ iff every finite Kripke structure $\mathcal{K}$ which is a finite tree satisfies A.

5- Construct a Kripke structure $\mathcal{K}$ over $\mathcal{Q}$, which is a denumerable tree and such that, for every propositional formula A over $\mathcal{Q}$,

$$\vdash_{\mathrm{LJ}} A \Leftrightarrow \mathcal{K} \mid\!\vdash\!- A.$$