Bordeaux university
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LOGICS

TD 2 : Cut elimination

Exercice 2.1 Harrop
1- Let us consider the theories (i.e. sets of formulas) : $\mathrm{EG}, \mathrm{P}_0, \mathrm{PA}, \mathrm{MO}, \mathrm{GR}$. Which ones are Harrop theories ? What (interesting) conclusions can we draw from this ?
Exercice 2.2 PA versus P_0
1- Let us denote by P'_0 (resp. PA') the set of formulas of P_0 (resp. PA) the axiom A2 excepted. Is PA' a Harrop theory ?
2- Give a derivation, within LK, of $\mathrm{PA}' \models A2$.
3- Can you give a derivation within LJ of $\mathrm{PA}' \models A2$? does there exist a term t, and a proof within LK of $\mathrm{PA}' \models \quad x = 0 \vee x = S(t)$?
Exercice 2.3 A non-standard model for P'_0.
Let $\mathcal{M} =_{\text{déf.}} \langle \mathcal{N}, 0_{\mathcal{N}}, \mathrm{S}_{\mathcal{N}}, +_{\mathcal{N}}, \times_{\mathcal{N}} \rangle$ the following « arithmetical » structure :

$$\begin{aligned}
\mathcal{N} &=_{\text{déf.}} (\mathbb{N} \times \{\bullet\}) \cup (\mathbb{N} \times \{\circ\}) \text{ where } \bullet \neq \circ \\
0_{\mathcal{N}} &=_{\text{déf.}} \langle 0, \bullet \rangle \\
\mathrm{S}_{\mathcal{N}} \langle p, \alpha \rangle &=_{\text{déf.}} \langle \mathrm{S}p, \alpha \rangle \text{ where } \alpha \in \{\bullet, \circ\} \\
\langle p, \alpha \rangle +_{\mathcal{N}} \langle q, \beta \rangle &=_{\text{déf.}} \langle p + q, \alpha \rangle \text{ where } \alpha, \beta \in \{\bullet, \circ\} \\
\langle p, \alpha \rangle \times_{\mathcal{N}} \langle q, \beta \rangle &=_{\text{déf.}} \langle p \times q, \beta \rangle \text{ where } \alpha, \beta \in \{\bullet, \circ\}
\end{aligned}$$

The domain of $\mathcal{N}$ consists of two copies of $\mathbb{N}$, the « black » integers $\langle p, \bullet \rangle$ and the « white » integers $\langle p, \circ \rangle$. The zero constant is interpreted by the black zero $\langle 0, \bullet \rangle$; the opérations successor, addition and multiplication are interpreted in such a way that :
— the successor keeps the color of its argument,
— the addition takes the la color of its *first* argument,
— the multiplication takes the color of its *second* argument.

1. Show that $\mathcal{M}$ is a model forP'_0. is it a model for P_0 ?
2. Show that none of the following properties is a logical consequence of P'_0 :

$$\forall x\ (0 + x = x) \tag{1}$$
$$\forall x, y\ (x + y = y + x) \tag{2}$$
$$\forall x, y, z\ (x + y = x + z \rightarrow y = z) \tag{3}$$
$$\forall x\ (x \times \mathrm{S}0 = x) \tag{4}$$
$$\forall x, y\ (x \times y = y \times x) \tag{5}$$

Exercice 2.4 Heyting arithmetics

Let Φ be some first-order formula over the signature of arithmetics.

1- Show that, if $\mathrm{P}'_0 \vdash \forall x \Phi$ is derivable within LJ, then $\mathrm{P}'_0 \vdash \Phi$ is derivable withinLJ.

2- does the property shown in question 1 remain true if we take PA as left-part of the sequent ? or if we still take the same sequent but consider the formal system LK ?

3- Show that, if $\mathrm{P}'_0 \vdash \forall x, \exists y \Phi(x, y)$ is derivable within LJ, then, there exists some terme t, such that $\mathrm{P}'_0 \vdash \Phi(x, t)$ is derivable within LJ.

4- Does the property of question 3 hold true if we take, as left-hand side of the sequent, PA ? or if we still take the same sequent but consider the formal system LK ?

5- Assume that $\mathrm{PA}' \vdash \forall x, \exists y \Phi(x, y)$ is derivable within LJ, by " only one recurrence" i.e.

$$\mathrm{P}'_0 \vdash_{\mathrm{LJ}} \exists y \ \ \Phi(0, y); \quad \mathrm{P}'_0 \vdash_{\mathrm{LJ}} (\exists y \Phi(x, y)) \to (\exists y \Phi(S(x), y))$$

5.1- Check that, under these assumptions, there does exist in LJ a derivation of $\mathrm{PA}' \vdash \forall x, \exists y \Phi(x, y)$.

For every $n \in \mathbb{N}$, we denote by $\underline{n}$ the term $S(S(\ldots(S(0))\ldots))$ which represents the integer n within the language of Peano arithmetics.

5.2- Show that, for every integer $n \in \mathbb{N}$, there edists a term t_n such that $\mathrm{P}'_0 \vdash_{\mathrm{LJ}} \Phi(\underline{n}, t_n)$.

5.3- Give an algorithm that computes the function $n \mapsto t_n$and which relies on the cut elimination algorithm.

Let us now admit that the property shown in exercice 4, question 5, is still valid for every formula of the form $\forall x, \exists y \Phi(x, y)$ (whether the derivation uses one reccurence or more).

Exercice 2.5 Recursive functions

We are thinking about the possibility of a converse of the above-admitted statement. Let us make the assumption(**ASSUMP**) for every computable, total, function$f : \mathbb{N} \to \mathbb{N}$, there exists a formula $\Phi(x, y)$ such that

(P1) $\mathrm{PA} \vdash \forall x, \exists_1 y \Phi(x, y)$ is derivable within LJ

where $\exists_1 x \ \ F(x)$ abbreviates $\exists x \ \ F(x) \wedge (\forall y, z \ \ F(y) \wedge F(z) \to y = z)$

(P2) for every integers n, m, $\mathbb{N} \models \Phi(\underline{n}, \underline{m})$ if and only if $f(n) = m$.

1- Can we deduce from this assumption an effective enumeration of all computable, total, functions ?

2- Find a diagonal argument showing that **ASSUMP** is false.

Let us admit the theorem [Matiyasevich, 1971] : a subset $M \subseteq \mathbb{N}$ is recursively enumerable iff, there exists an integer $q \in \mathbb{N}$ and a polynomial $P \in \mathbb{Z}[X, Y_1, \ldots, Y_q]$ such that, for every $x \in \mathbb{N}$,

$$x \in M \Leftrightarrow \exists \vec{y} \in \mathbb{N}^q, P(x, \vec{y}) = 0.$$

Exercice 2.6 PA is undecidable

Let q be some natural integer and P be a polynomial in $\mathbb{Z}[X, Y_1, \ldots, Y_q]$.

1- Show that, if $\mathbb{N} \models \exists \vec{y} P(\underline{n}, \vec{y}) = 0$.

then,there exists some natural integers vector $\vec{m}$ such that $\mathrm{PA} \vdash_{\mathrm{LK}} P(\underline{n}, \underline{\vec{m}}) = 0$.

2- Show that, if $\mathrm{PA} \vdash_{\mathrm{LK}} \exists \vec{y} \ \ P(\underline{n}, \vec{y}) = 0$, then $\mathbb{N} \models \exists \vec{y} P(\underline{n}, \vec{y}) = 0$.

3- Show that there exists a polynomial P such that the problem

Instance : $n \in \mathbb{N}$; Question : does there exist a vector $\vec{m} \in \mathbb{N}^q$ such that $P(n, \vec{m}) = 0$?

is undecidable.

4- Show that the following problem is undecidable too :

Instance : a formula Φ ; Question : $\mathrm{PA} \vdash_{\mathrm{LK}} \Phi$?