# Indecomposable permutations with a given number of cycles

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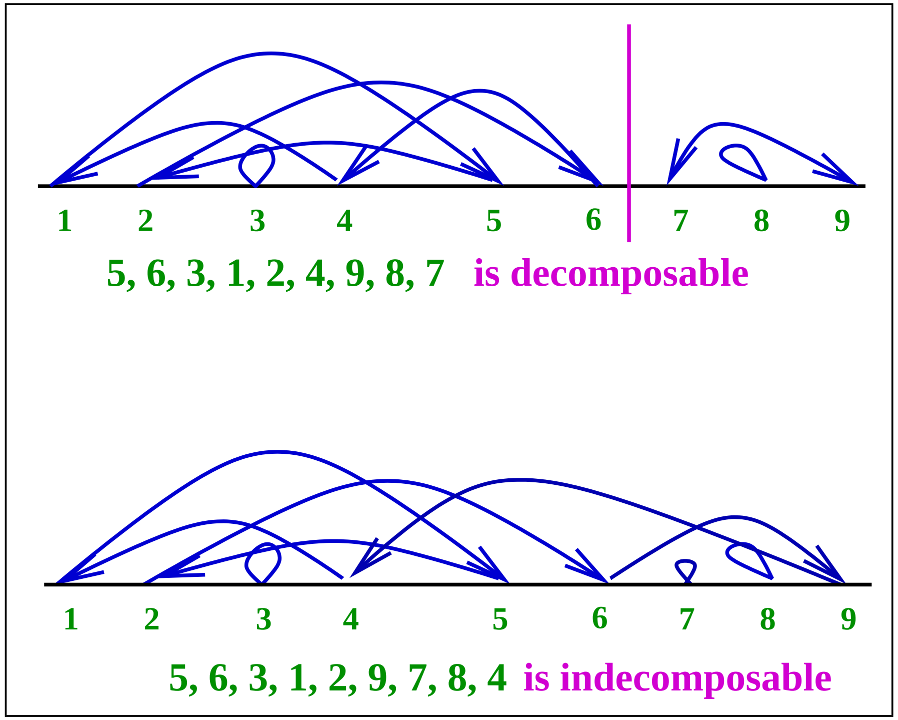
- **Part I:** **Permutations.**
- **Part II:** **Matchings** (also called Fixed point Free Involutions)

# Part 1: Indecomposability of (general) permutations

A permutation $a_1, a_2, \ldots, a_n$ is *decomposable* if there exists $p < n$ such that $a_1, a_2, \ldots, a_p$ is a permutation of $1, 2, \ldots, p$



# Almost all permutations are indecomposable

The number of permutations of $\mathcal{S}_n$ having $m$ cycles is the Stirling number $s_{n,m}$.

Let $c_n$ be the number of indecomposable permutations of $\mathcal{S}_n$ and $c_{n,m}$ be the number of those of $\mathcal{S}_n$ having $k$ cycles, then:

**Theorem** (Comtet, 1969)

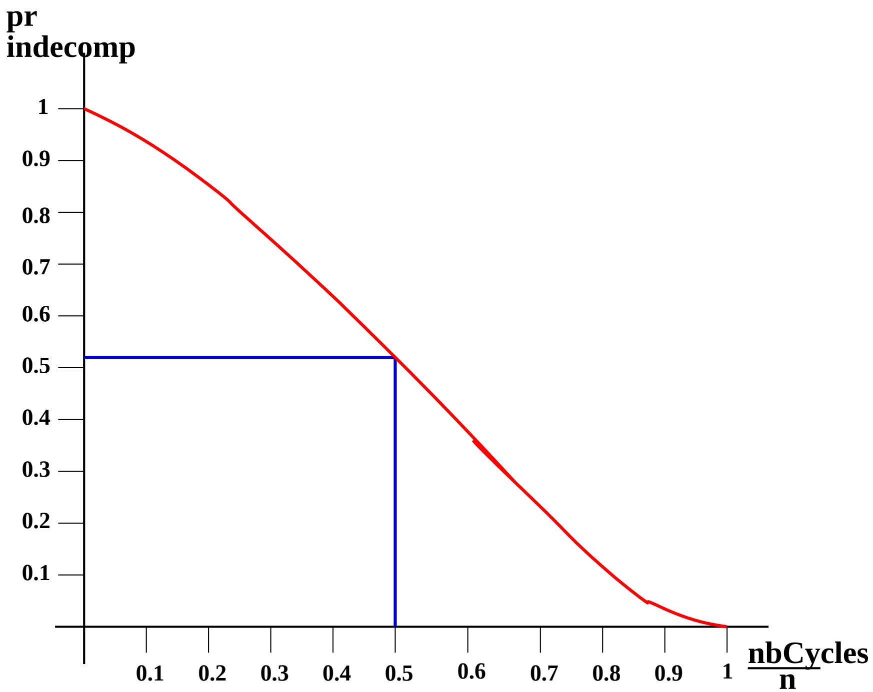
$$\frac{c_n}{n!} = 1 - \frac{2}{n} + O(\frac{1}{n^2})$$

The probability depends heavily on the number of cycles

- Permutations with 1 cycle: $\frac{c_{n,1}}{s_{n,1}} = 1$
- Permutations with $n-1$ cycles $\frac{c_{n,n-1}}{s_{n,n-1}} = \frac{2}{n(n-1)}$

# Main results

The shape of the curve $\frac{c_{n,m}}{s_{n,m}}$ as a function of $\frac{m}{n}$ when $n$ and $m$ tend to $\infty$



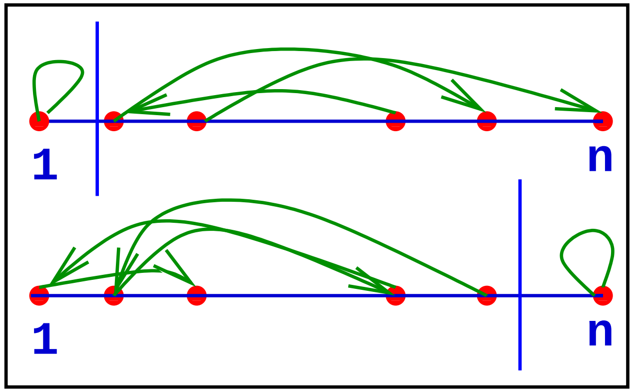

About $51.1\ \%$ of the permutations of $\mathcal{S}_n$ with $\frac{n}{2}$ cycles are indecomposable

# Sketch of the proof (I/II)

**Lemma 1** If the following condition holds, then the permutation $a_1, \ldots a_n$ is decomposable:

$$a_1 = 1 \text{ or } a_n = n$$

Moreover when $n$ and $m$ are sufficiently large almost all decomposable permutations **with** $m$ **cycles** satisfy this condition.




The number of permutations of $\mathcal{S}_n$ with $m$ cycles satisfying the condition is :

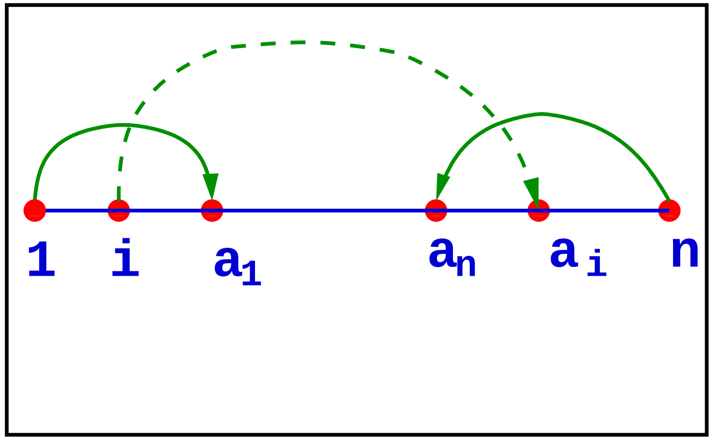
$$2s_{n,m-1} - s_{n-2,m-2}$$

# Sketch of the proof (II/II)

**Lemma 2** If the following condition holds, then the permutation $a_1, \dots a_n$ is indecomposable:

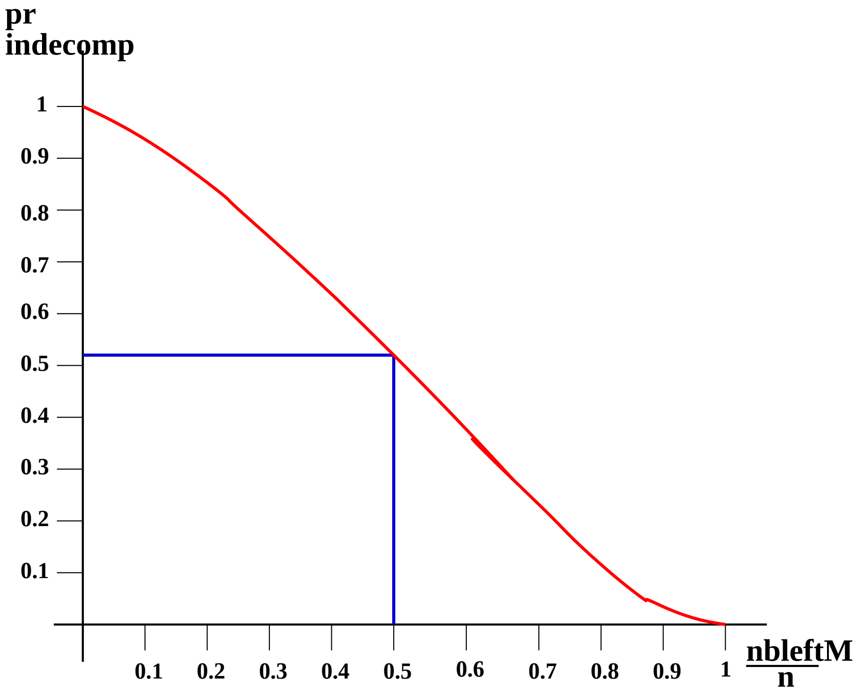
$$\exists i, i \leq a_1 \text{ and } a_i > a_n$$

Moreover when $n$ and $m$ are sufficiently large almost all indecomposable permutations **with** $m$ **cycles** satisfy this condition.



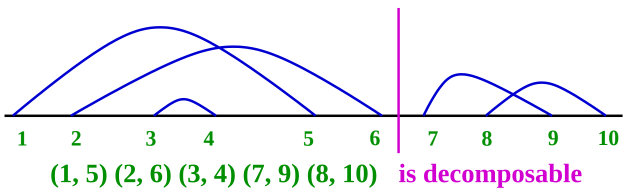
# Permutations with $m$ left-to-right maxima

**Remark** The probability for a permutation of $S_n$ with $m$ left-to-right maxima to to be decomposable is the same as the probabilty for permutations with $m$ cycles to be indecomposable

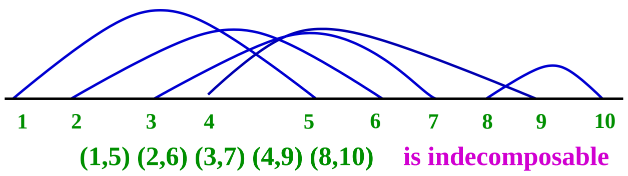


# Part 2: Indecomposability of Matchings

**Definition** A matching is a permutation in which all cycles have length $2$. Matchings are among the permutations which have $m = \frac{n}{2}$ cycles.




**Lemma** About $1 - \frac{2}{n} + O(\frac{1}{n^2})$ matchings are indecomposable



# Maps and matchings

| | | |
|---|---|---|
| Indecomposable Matchings | $\longleftrightarrow$ | Rooted maps on Orientable Surfaces |
| # of cycles of the matching -1 | $\longleftrightarrow$ | # of edges of the map |
| # of left to right maxima | $\longleftrightarrow$ | # of vertices of the map |

**Question:** Among matchings with $k$ left-to-right maxima how many are indecomposable ?

# Maps with 2 edges and indecomposable matchings with 3 cycles

Indecomposable matchings with 1 left-to-right maximum:

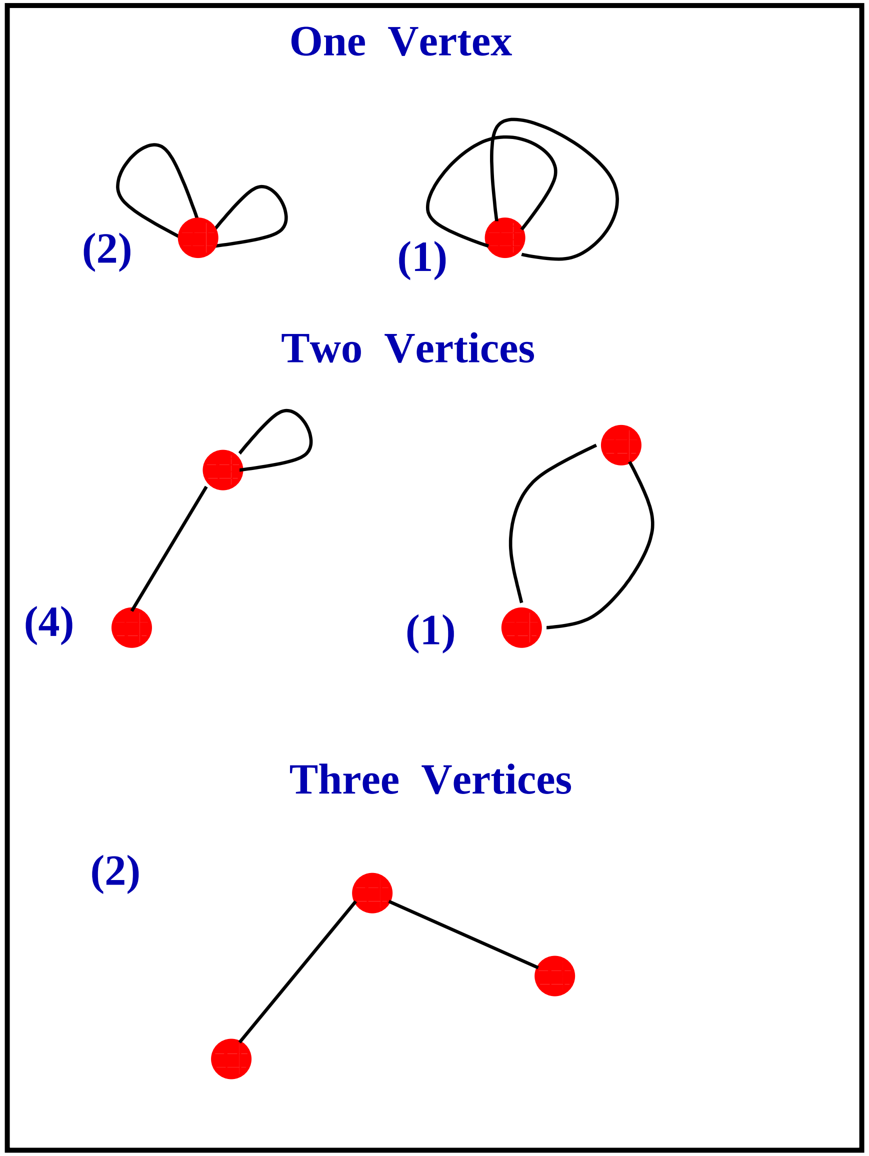
$(1,6)(2,3)(4,5)$
$(1,6)(2,4)(3,5)$
$(1,6)(2,5)(3,4)$

Indecomposable matchings with 2 left-to-right maxima:

$(1,5)(2,4)(3,6)$, $(1,4)(2,6)(3,5)$
$(1,5)(2,3)(4,6)$, $(1,3)(2,6)(4,5)$
$(1,5)(2,6)(3,4)$

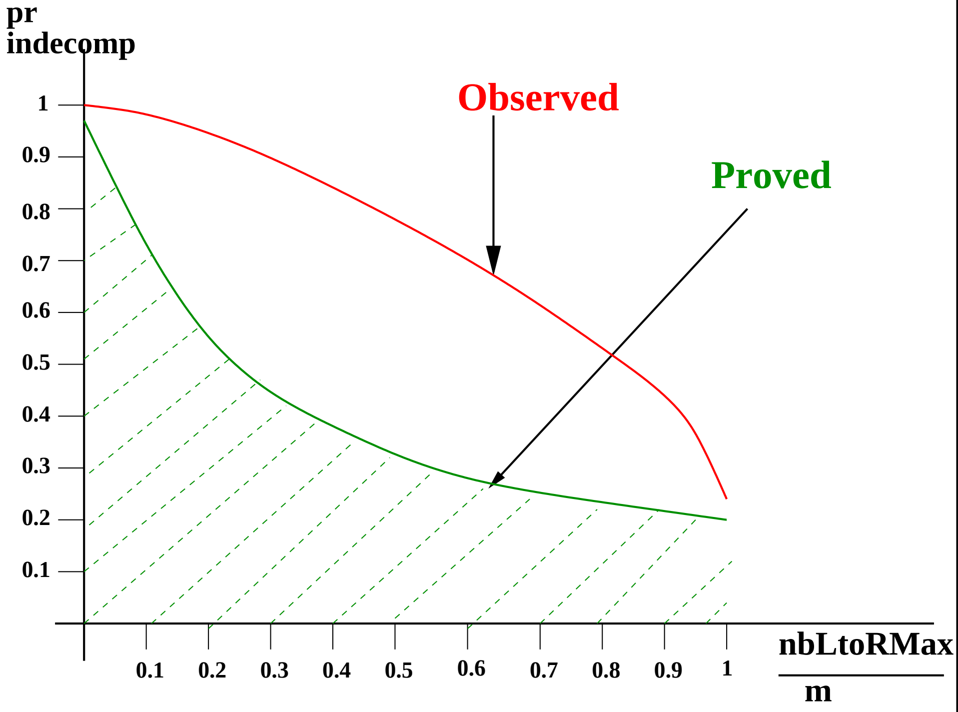
Indecomposable matchings with 3 left-to-right maxima.

$(1,4)(2,5)(3,6)$
$(1,3)(2,5)(4,6)$



# Result for matchings

**Theorem 2** The probability to be indecomposable for a matching with $m$ cycles and $k$ left-to-right maxima is greater than $\frac{m}{m+4k}$

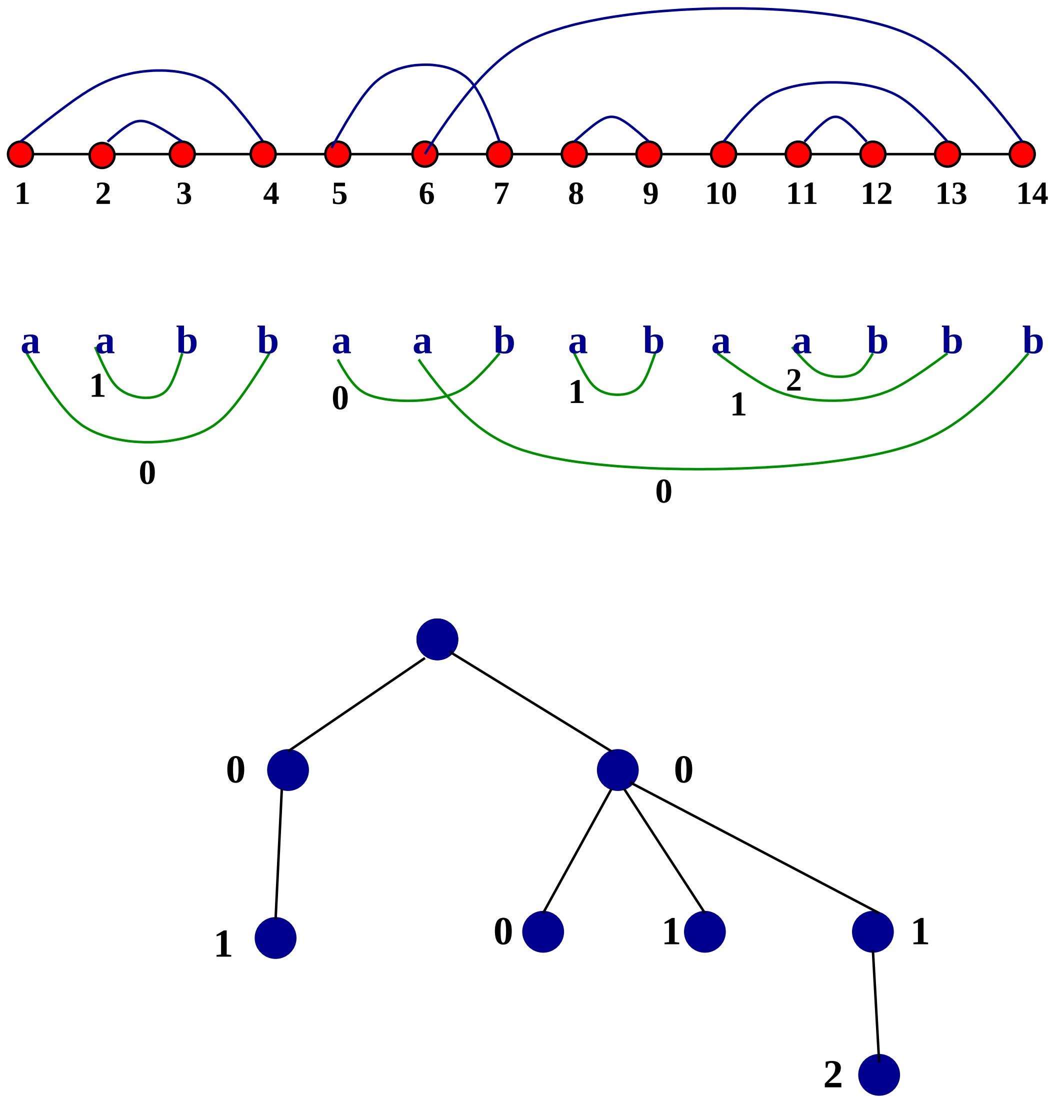


# Bijections

| | | |
|---|---|---|
| Matchings | $\longleftrightarrow$ | Labelled trees |
| Indecomposable matching | $\longleftrightarrow$ | root has only one child |
| # of left to right maxima | $\longleftrightarrow$ | # of nodes labeled 0 |

**Labels** The nodes are labeled by non negative integers less than their distances from the root

# Bijection : Matchings, Dyck words, Trees



# Sketch of the proof of Theorem 2 (I/II)

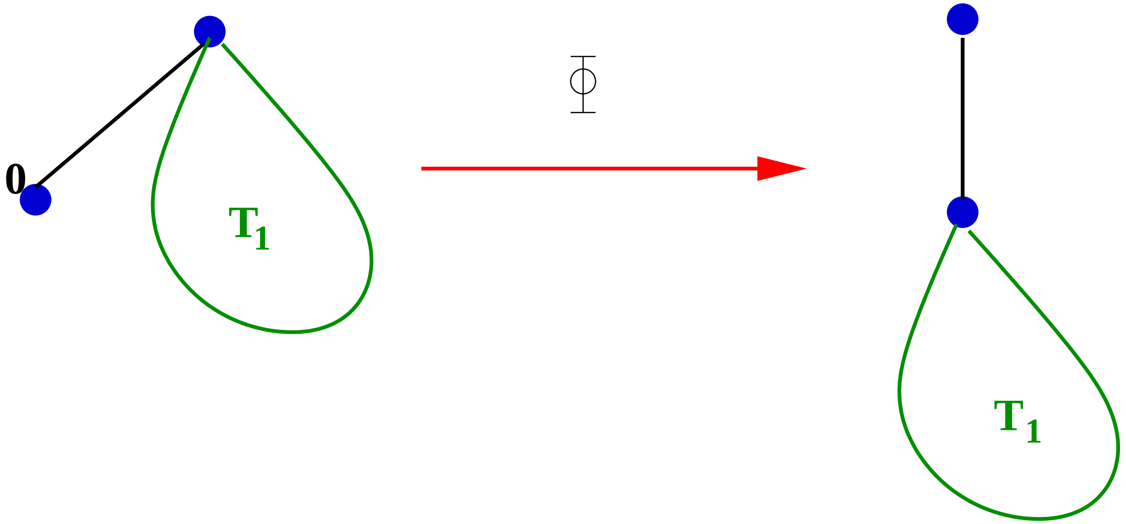
**Lemma 3** There exists a mapping $\Phi$

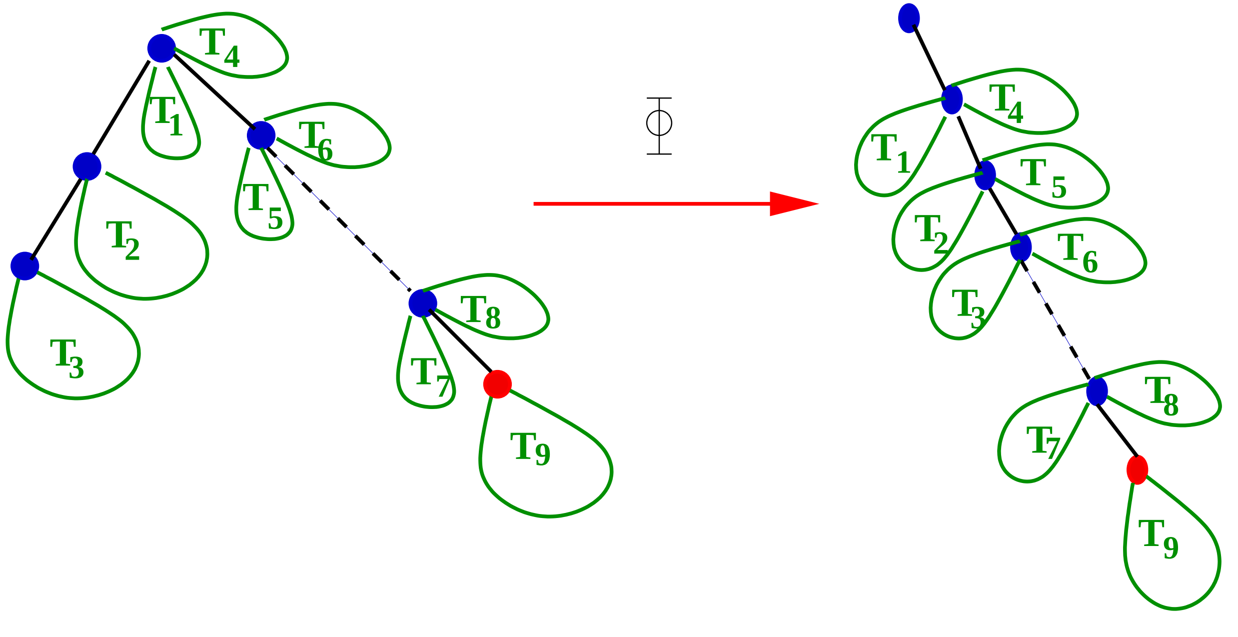
- **From:** the set of decomposable trees with $k$ nodes labeled $0$, all other nodes unlabeled and where a node not in the first subtree (below the root) is marked
- **To:** the set of indecomposable trees with $k$ nodes labeled $0$ and a marked node.

- An indecomposable tree has a number of inverse images by $\Phi$ not greater than $2$
- Denote $h_1, \ldots, h_{m-k}$ the distances from the root of the unlabeled nodes of a decomposable tree $T$ then those $h'_1, \ldots h'_{m-k}$ of $\Phi(T)$are such that :

$$h'_i \geq h_i + 1$$

# Mapping Φ



## Sketch of the proof of Theorem 2 (II/II)

**Lemma 4** The number $L$ of labelings of a tree $T$ with $k$ vertices labeled $0$ in which the distances from the root to the unlabeled vertices are $h_1, h_2, \ldots, h_{m-k}$, and that $L'$ of labelings of a tree $T'$ in which these distances are $h'_1, h'_2, \ldots, h'_{m-k}$, with $h'_i \geq h_i + 1$ satisfy:

$$L = h_1 h_2 \cdots h_{m-k}, \quad L' = h'_1 h'_2 \cdots h'_{m-k}, \quad L \leq \frac{k}{m} L'$$