Parking Functions and the Cyclic Lemma

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1 Introduction

Parking functions where introduced and enumerated by Konheim and Wiess [3], bijective proofs of these formulas were obtained in different ways in [4], [2], [5]. More recently Stanley and Pitman (in [6]) generalized this notion by introducing what is called $(a, \overline{b})$-parking functions. Some enumrative results were also obtained (in [1] for instance) on them using a bijection with families of trees. We give here another proof of the enumeration formulas for $(a, \overline{b})$-parking functions by using the Cyclic Lemma for words.

A formal definition is the following: an $(a, \overline{b})$-parking function of length n is a sequence $\mathbf{p} = p_1, p_2, \ldots, p_n$ of non negative integers less than $a + bn$ such that there exists a permutation $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ of $\mathfrak{S}_n$ such that, for all i we have, $p_i < (\alpha_i - 1)b + a$. It is often said that such permutation is a *certificate* for the sequence $\mathbf{p}$. For instance, $7, 0, 4$ is a $(4, \overline{2})$-parking function, for which $3, 1, 2$ is a certificate; but $3, 7, 6$ is not.

It is easy to check that any sequence obtained by permuting the elements of an $(a, \overline{b})$-parking function is also an $(a, \overline{b})$- parking function. Hence an easy way to check that a sequence $\mathbf{p} = p_1, p_2, \ldots p_n$ is an $(a, \overline{b})$- parking function consists in reordering $\mathbf{p}$ such that to obtain a weakly increasing sequence: $\mathbf{p}' = p'_1 \leq p'_2 \ldots \leq p'_i \ldots \leq p'_n$ then in checking that for any i one has: $p'_i \ < \ (i-1)b + a$

2 Main results

Let $\Sigma_{n,m}$ be the set of sequences of non negative numbers of length n with all elements less than m, hence the number of elements in $\Sigma_{n,m}$ is m^n. In the sequel we will consider $(a, \overline{b})$-parking functions of length n and we will denote $m = a + nb$.

Definition *Let $\mathbf{u} = u_1, u_2, \ldots, u_n$ be a sequence in $\Sigma_{n,m}$, a conjugate of $\mathbf{u}$ is a sequence $\mathbf{v} = v_1, v_2, \ldots, v_n$ such that there exists an integer k for which each v_i is obtained from u_i by adding $k \mod m$. More formally if for $i = 1, n$ we have:*

$$v_i \ = \ u_i + k \qquad \mod m$$

It is clear that any sequence of $\Sigma_{n,m}$ has exactly m different conjugates.

The main result of this note is the following statement which generalizes a well known result concerning parking functions:

Theorem 1 *For any sequence u of $\Sigma_{n,m}$ where $m = a + nb$ the number of conjugates of u which are $(a, \overline{b})$-parking functions is equal to a.*

Proof

See section 4.

□

EXAMPLE For $a = 4, b = 2, n = 3$ we have $m = 10$ and the 10 conjugates of $(0, 3, 7)$ are the following sequences :

$$((1, 4, 8), (2, 5, 9), (3, 6, 0), (4, 7, 1), (5, 8, 2), (6, 9, 3), (7, 0, 4), (8, 0, 5), (9, 1, 6), (0, 3, 7).$$

among them only $(0, 3, 7)$, $(3, 6, 0)$, $(4, 7, 1)$ and $(7, 0, 4)$ are $(4, 2)$-parking functions. ♢

Corollary 1 *The number of $(a, \overline{b})$-parking functions of $\Sigma_{n,m}$ is*

$$a(a + nb)^{n-1}$$

Proof

The number of conjugacy classes of $\Sigma_{n,m}$ is

$$\frac{m^n}{m} = m^{n-1}$$

and in each class the number of $(a, \overline{b})$-parking functions is a. □

A sequence $\mathbf{u} = u_1, u_2, \ldots, u_n$ is symmetric if it is equal to its mirror image, more precisely if for all i one has: $u_i = u_{n-i+1}$. We have also:

Corollary 2 *The number of symmetric $(a, \overline{b})$-parking functions of $\Sigma_{n,m}$ for $n = 2k$ is $a(a + nb)^{k-1}$, this number for $n = 2k + 1$ is $a(a + nb)^k$.*

Proof

It suffices to notice that the number symmetric sequences of length $2k$ and $2k + 1$ are m^k and m^{k+1} respectively, moreover all the conjugates of a symmetric sequence are symmetric. □

3 Coding weakly increasing sequences by words

We consider words, these are sequences of letters, on the alphabet:

$$X = \overline{x}, x_1, x_2, \ldots x_p, \ldots$$

The empty word is denoted ε. For a word $f = f_1 f_2 \ldots f_k$ of length k ($f_i \in X$) we denote $\delta(f) = \sum_{i=1}^{k} \delta(f_i)$ where $\delta(\overline{x}) = -1$ and $\delta(x_p) = p$.

Definition *A word f on the alphabet X is a Lukasiewicz word if $\delta(f) < 0$ and if for any f' such that $f = f'f$", $f' \neq f$ one has:*

$$\delta(f') > \delta(f).$$

The factorizations of a word f are the pairs f of words $(f', f")$ such that $f = f'f"$ and $f" \neq \varepsilon$; hence a word of length n has exactly n factorizations. Notice that for such factorizations the words $f"f'$ are not necessarily all different. However the *Cyclic Lemma* states that each word f on the alphabet X, such that $\delta(f) < 0$, has exactly $\delta(f)$ factorizations $(f', f")$ for which $f"f'$ is a Lukaciewicz word (see for instance [?] chap. 11, Theorem 3.6).

For any weakly increasing sequence $\mathbf{u} = u_1, u_2, \ldots, u_n$ of $\Sigma_{n,m}$ (i.e. such that $u_1 \leq u_2 \leq \ldots \leq u_n$) let $f = \Phi(u)$ be the word on the alphabet $x, \overline{x}$ given by

$$f = \overline{x}^{u_1}\ x\ \overline{x}^{t_1} x\ \overline{x}^{t_2} \ldots x\ \overline{x}^{t_{n-1}} x\ \overline{x}^{t_n}$$

where for $i < n$, $t_i = u_{i+1} - u_i$ and $t_n = m - u_n$. In this notation $\overline{x}^j$ means a sequence of j consecutive letters $\overline{x}$.

Proposition 1 *The conjugates of the sequence* $\mathbf{u}$ *correspond to the words* $f"f'$ *obtained from the factorizations* $(f', f")$ *of* $f = \Phi(\mathbf{u})(= f'f")$*, for which* f' *ends with* $\overline{x}$*.*

Proof

It suffices to notice that if $f = \Phi(\mathbf{u})$ then u_i is equal to the number of occurrences of $\overline{x}$ preceding the i-th occurrence of x in f. Hence, if $f = f'\overline{x}g\overline{x}$ where g contains no occurrences of $\overline{x}$ then the sequence $\mathbf{v}$ such that $\Phi(\mathbf{v}) = g\overline{x}f'\overline{x}$ is obtained from $\mathbf{u}$ by adding $1 \mod m$ to all the u_i.

□

EXAMPLE For $a = 4, b = 2$, let us return to the sequence $(0, 3, 7)$ considered above, we have:

$$\Phi(0, 3, 7) = x\overline{x}\overline{x}\overline{x}x\overline{x}\overline{x}\overline{x}\overline{x}x\overline{x}\overline{x}\overline{x}$$

which has 10 factorizations such that f' ends with $\overline{x}$ each giving rise to one of the 10 sequences obtained above. ◇

4 Proof of Theorem 1.

It suffices in order to prove this Theorem to consider weakly increasing sequences, since the condition for being an $(a, \overline{b})$-parking function is invariant by permutation of the elements.

Let us modify the word $f = \Phi(\mathbf{u})$ replacing each occurrence of x by x_b, denote g the word obtained in such way.

We prove that g is a Lukaciewicz word if and only if the sequence $\mathbf{u}$ is an $(a, \overline{b})$-parking function. In that case g has n occurrences of x_b and m occurrences of $\overline{x}$ hence $\delta(g) = nb - m = -a$. In order to prove that g is a Lukaciewicz word we have to prove that for any prefix g' of g, $\delta(g')$ is greater than $\delta(g)$. It suffices to considered those prefixes of g' followed by an occurrence of x_b. But if g' is followed by the i-th occurrence of x_b it contains exactly u_i occurrences of $\overline{x}$ and $\delta(g') = b(i-1) - u_i$, $\delta(g') > -a$ is equivalent to $u_i < b(i-1) + a$ which is exactly the condition for la condition pour $\mathbf{u}$ to be an

$(a,\overline{b})$-parking function. Theorem 1 then follows from the Cyclic Lemma and Proposition 1. □

Example The 4 factorizations of the word

$$f = x_2\overline{xxx}x_2\overline{xxxx}x_2\overline{xxx}$$

which give rise to a Lukaciewicz word are f and the three other ones:

$$x_2\overline{xxx}x_2\overline{xxx}x_2\overline{xxxx},\quad \overline{x}x_2\overline{xxx}x_2\overline{xxx}x_2\overline{xxx},\quad x_2\overline{xxxx}x_2\overline{xxx}x_2\overline{xxx}$$

corresponding to the 4 sequences obtained above. ♢

References

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