Invariant and covariant polynomials for the (generalized) symmetric group

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Gröbner bases.

$X_n = x_1, x_2, \ldots, x_n$

Graded lexicographic order in $\mathbb{C}[X_n]$

Example 1

$$P = 2x_1^3\,x_2 - x_1\,x_2\,x_3^2 + x_2^4 - 3x_2\,x_3^2\ , \quad LM(P) = x_1^3\,x_2$$

Definition 2 A set $\mathcal{G} = \{g_1, \ldots, g_m\} \subset I$ is a Gröbner basis if:

$$\forall P \in I\ , \quad \exists g_i \in \mathcal{G}\ /\ LM(g_i) \,|\, LM(P).$$

Proposition 3 *Let* $\{g_1, \ldots, g_m\}$ *be a Gröbner basis of* I*. Then*

$$\mathcal{B} = \{X_n^{\mathbf{p}}\ /\ \forall i,\ LM(g_i) \not|\ X_n^{\mathbf{p}}\}$$

is a basis of the quotient $\mathbb{C}[X_n]/I$*.*

Proof.

• **Spanning set:**

We prove that every monomial $X_n^{\mathbf{q}} \notin \mathcal{B}$ may be rewritten (modulo I) with smaller monomials.

$$\begin{aligned} \exists i,\ X_n^{\mathbf{q}} &= X_n^{\alpha}\, LM(g_i) \\ X_n^{\mathbf{q}} &= X_n^{\alpha}\,(LM(g_i) - g_i) + X_n^{\alpha}\, g_i \\ &\equiv X_n^{\alpha}\,(LM(g_i) - g_i) \quad [I] \end{aligned}$$

• **Independent set:**

If $\exists(\alpha_{\mathbf{p}})$ such that

$$\begin{aligned} \sum \alpha_{\mathbf{p}}\, X_n^{\mathbf{p}} &\equiv 0 \quad [I] \\ \sum \alpha_{\mathbf{p}}\, X_n^{\mathbf{p}} &\in I \\ LM(\sum \alpha_{\mathbf{p}}\, X_n^{\mathbf{p}}) &\in LM(I) \end{aligned}$$

Syzygies

$F = \{f_1, f_2\}$ with
$f_1 = x_1^3 + 2x_1x_2^2 + x_2^3 \quad \text{and} \quad f_2 = x_1x_2 - x_2^2$

$$
\begin{aligned}
S(f_1, f_2) &= x_2\, f_1 - x_1^2\, f_2 \\
&= x_2(x_1^3 + 2x_1x_2^2 + x_2^3) \\
&\quad -x_1^2(x_1x_2 - x_2^2) \\
&= 2x_1x_2^3 + x_2^4 - x_1^2x_2^2 \\
&= -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\
&\equiv -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\
&\quad +x_1x_2(x_1x_2 - x_2^2) \\
&= x_1x_2^3 + x_2^4 \\
&\equiv x_1x_2^3 + x_2^4 - x_2^2(x_1x_2 - x_2^2) \\
&= 2x_2^4 \\
&= \overline{S(f_1, f_2)}^F
\end{aligned}
$$

Buchberger's criterion

Theorem 4 *A set $\mathcal{G} = \{g_1, \ldots, g_m\}$ is a Gröbner basis if and only if*

$$\forall i \neq j, \quad \overline{S(g_i, g_j)}^{\mathcal{G}} = 0.$$

Proposition 5 *If f and g have* disjoint *leading monomials then*

$$\overline{S(f, g)}^{\{f,g\}} = 0.$$

$\mathcal{S}_n$: symmetric group
group of permutations of $\{1, \ldots, n\}$.

$$\sharp\, \mathcal{S}_n = n!$$

Classical action of $\mathcal{S}_n$ on $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[X_n]$:

$$\sigma \,.\, P(x_1, \ldots, x_n) = P(X_n.\sigma) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

$P \in \mathbb{C}[X_n]$ is *symmetric* iff

$$\forall \sigma \in \mathcal{S}_n,\ \sigma \,.\, P = P.$$

Sym_n : space of symmetric polynomials.

Bases:

$$\begin{aligned} p_2(X_3) &= x_1^2 + x_2^2 + x_3^2 \\ e_2(X_3) &= x_1x_2 + x_1x_3 + x_2x_3 \\ h_2(X_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \end{aligned}$$

Definition 6 Harmonic polynomials :

$$\begin{aligned} \mathbf{H}_n &= \{P \in \mathbb{C}[X_n] \ / \ \forall k > 0,\ e_k(\partial)\, P = 0\} \\ &\simeq \mathbb{C}[X_n]/\langle e_k,\ k > 0\rangle. \end{aligned}$$

Theorem 7 (Artin)

$$\dim \mathbf{H}_n = n!\,.$$

Proof. (Gröbner bases) ($n = 4$)

$$
\begin{aligned}
h_1(X_4) &= x_1 + x_2 + x_3 + x_4 \\
h_2(X_4) &= x_1\, h_1(X_4) + h_2(x_2, x_3, x_4) \\
&\equiv h_2(x_2, x_3, x_4) \\
h_3(X_4) &= x_1\, h_2(X_4) + h_3(x_2, x_3, x_4) \\
&\equiv h_3(x_2, x_3, x_4) \\
&= x_2\, h_2(x_2, x_3, x_4) + h_3(x_3, x_4) \\
&\equiv h_3(x_3, x_4) \\
h_4(X_4) &= x_1\, h_3(X_4) + h_4(x_2, x_3, x_4) \\
&\equiv h_4(x_2, x_3, x_4) \\
&= x_2\, h_3(x_2, x_3, x_4) + h_3(x_3, x_4) \\
&\equiv h_4(x_3, x_4) \\
&= x_3\, h_3(x_3, x_4) + h_4(x_4) \\
&\equiv h_4(x_4)
\end{aligned}
$$

Leading Monomials: x_1, x_2^2, x_3^3, x_4^4.

In general: $x_1, x_2^2, \ldots, x_n^n$.

This is a Gröbner basis of $\langle e_k,\ k > 0 \rangle$.

We get a basis of the quotient :

$$\mathcal{A}_n = \{X_n^\epsilon\ ,\quad 0 \le \epsilon_i \le i-1\}$$

(*Artin monomials*)

Corollary 8 *Hilbert series:*

$$\begin{aligned} F_q(\mathbf{H}_n) &= \sum_k q^k \dim \pi_k(\mathbf{H}_n) \\ &= 1(1+q)\cdots(1+q+\cdots+q^{n-1}) \\ &= [n!]_q \end{aligned}$$

Quasi-symmetrizing action of S_n:

Example 9 $n = 5$

$$\begin{aligned} & (1,4) \bullet (x_1^2\, x_2\, x_5) \\ = \quad & (1,4) \bullet \{x_1, x_2, x_5\}^{(2,1,1)} \\ = \quad & \{x_4, x_2, x_5\}_<^{(2,1,1)} \\ = \quad & \{x_2, x_4, x_5\}^{(2,1,1)} \\ = \quad & x_2^2\, x_4\, x_5. \end{aligned}$$

Invariants: $Qsym_n \supset Sym_n$

$$x_1^2x_2 + x_1^2x_3 + x_2^2x_3 \in QSym_3 \backslash Sym_3.$$

[2,1,0]+[2,0,1]+[0,2,1]

Definition 10 $\alpha = (\alpha_1, \ldots, \alpha_l)$ a composition of $|\alpha| = k \geq 0$,

$$M_\alpha(X_n) = \sum_{0 \leq i_1 < \cdots < i_l \leq n} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

and

$$F_\alpha(X_n) = \sum_{\beta \succeq \alpha} M_\beta(X_n).$$

Example 11 $n = 3, \alpha = (1,2)$:

$$\begin{aligned} M_\alpha &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \\ F_\alpha &= M_{(1,2)} + M_{(1,1,1)} \\ &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3 \end{aligned}$$

Example 12

$$F_{(3,2)} = M_{(3,2)} + M_{(1,2,2)} + M_{(2,1,2)} + M_{(1,1,1,2)}$$

$$+ M_{(3,1,1)} + M_{(1,2,1,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}.$$

Super-harmonic polynomials

$$\mathcal{I}_n = \langle F_\alpha,\ |\alpha| > 0 \rangle.$$

Definition 13 We define

$$Q_n = \mathbb{C}[X_n]/\mathcal{I}_n,$$

and the *super-harmonic space*

$$\mathbf{SH}_n = \{P \in \mathbb{C}[X_n] \ / \ \forall |\alpha| > 0,\ M_\alpha(\partial)P = 0\}.$$

Remark 14

$$Sym_n \subset QSym_n \implies \mathbf{SH}_n \subset \mathbf{H}_n.$$

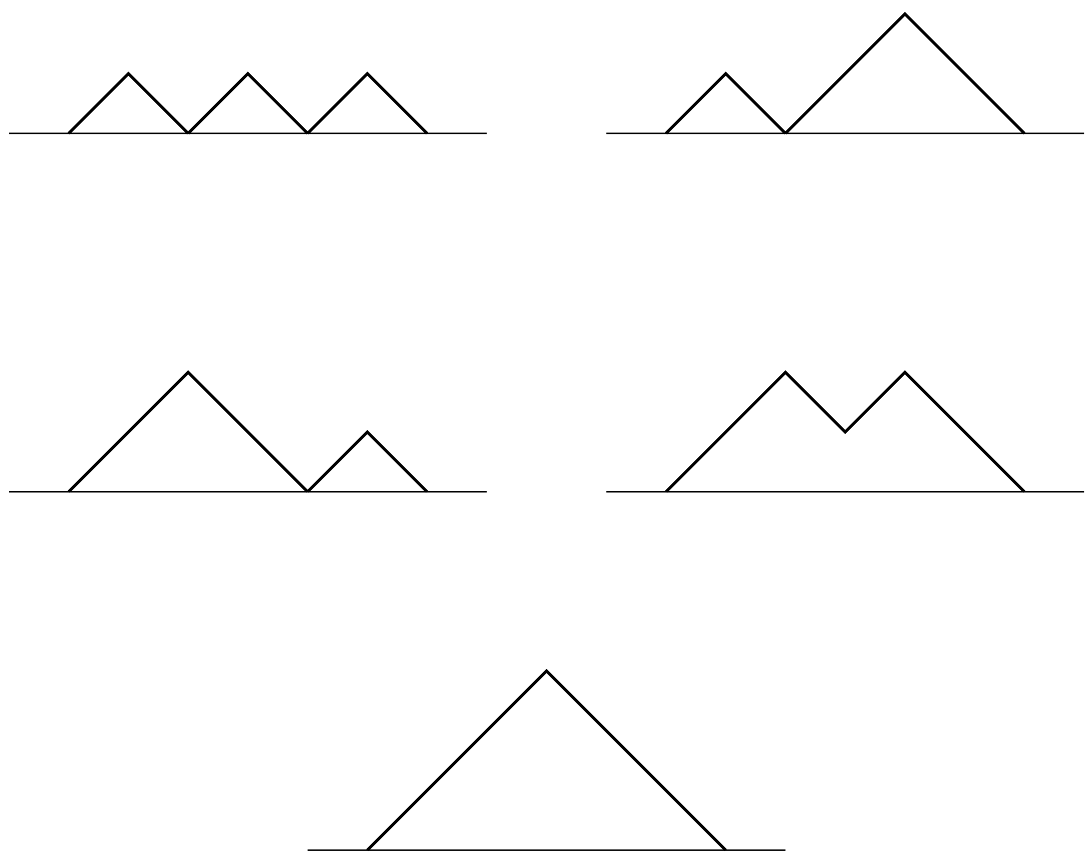
Catalan numbers

Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

count (among other things !) Dyck paths.

Example 15 $n = 3,\ C_3 = 5$



Theorem 16 (Aval, Bergeron, Bergeron) *The dimension of the super-harmonic space is*

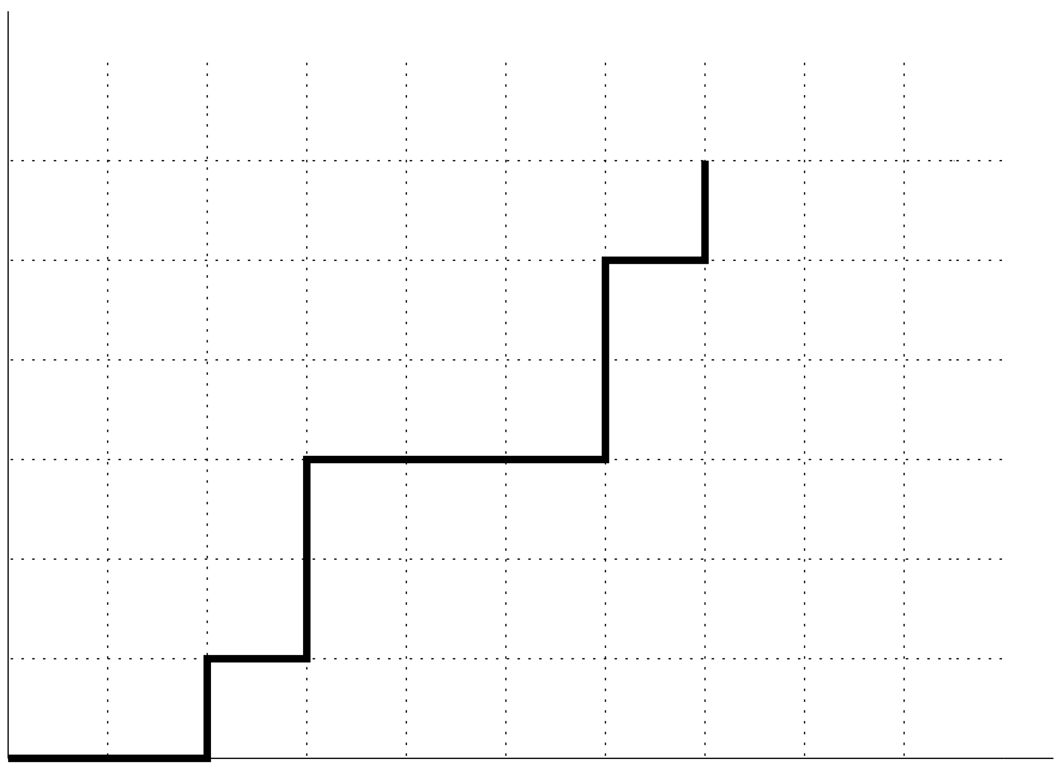
$$\dim \mathbf{SH}_n = \dim Q_n = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Coding vectors in $\mathbb{N}^n$ by plane paths

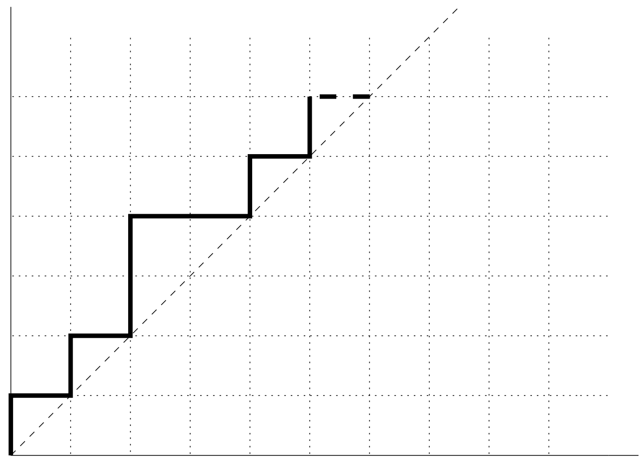
Example 17 For $n = 6$, we associate to

$$\epsilon = (2, 1, 0, 3, 0, 1)$$

the path P_ϵ:

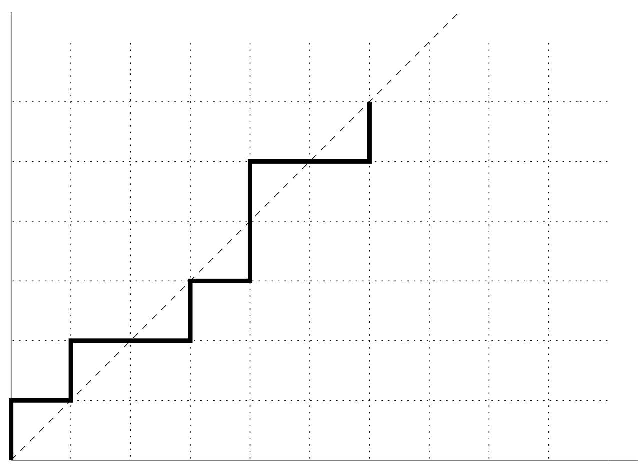


Dyck path:



$$\epsilon = (0, 1, 1, 0, 2, 1)$$

Transdiagonal path:



$$\epsilon = (0, 1, 2, 1, 0, 2)$$

Theorem 18 *The set of monomials*

$$\{X_n^\eta \ / \ \eta \ Dyck\}$$

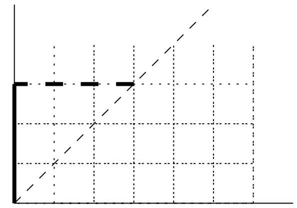
is a basis for Q_n*.*

Example 19 $(n = 3)$

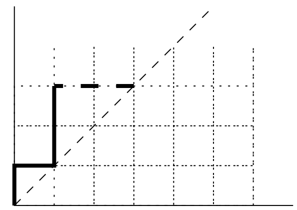
$$
\begin{aligned}
F_1 &= x_1 + x_2 + x_3 \\
F_2 &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\
&= x_1(x_1 + x_2 + x_3) + x_2^2 + x_2x_3 + x_3^2 \\
&\equiv x_2^2 + x_2x_3 + x_3^2 \\
F_{12} &= x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 \\
&= x_1(x_2^2 + x_2x_3 + x_3^2) + x_2x_3^2 \\
&\equiv x_2x_3^2 \\
F_3 &= x_1F_2(x_1, x_2, x_3) + F_3(x_2, x_3) \\
&\equiv x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3 \\
&\equiv x_2(x_2^2 + x_2x_3 + x_3^2) + x_3^3 \\
&\equiv x_3^3
\end{aligned}
$$

The monomials which are not multiple of one of the $\{x_1, x_2^2, x_2 x_3^2, x_3^3\}$ are:

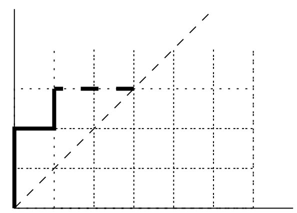
1



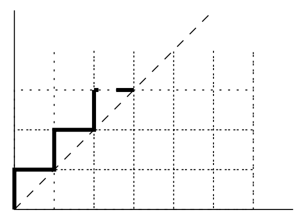
x_2



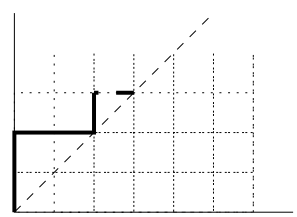
x_3



$x_2 x_3$



x_3^2



Proof.

Construction of a Gröbner basis of $\mathcal{I}_n$

We construct a set of polynomials

$$\mathcal{G} = \{G_\epsilon \in Q[X_n] \ ; \ \epsilon \text{ transdiagonal}\} \subset \mathcal{SI}_n$$

by induction on the length of ϵ:

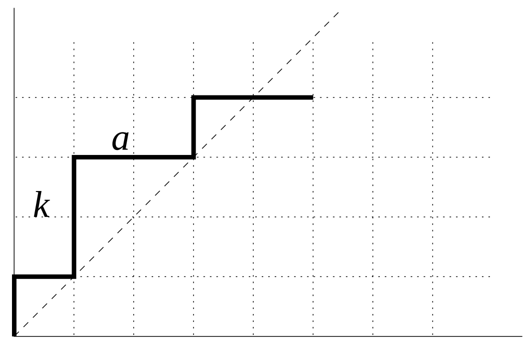
- if $\epsilon = \alpha 0^*$, then $G_\epsilon = F_\alpha$;
- otherwise, we look at the rightmost zero:

  $$\epsilon = \eta 0 a \beta 0^* = \eta 0 a \beta,$$

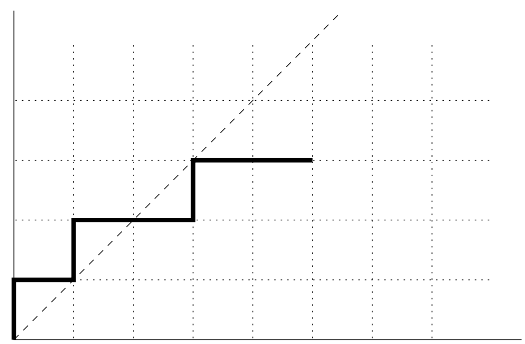
  and we set (for 0 in place k):

  $$G_\epsilon = G_{\eta a \beta} - x_k G_{\eta (a-1) \beta}.$$

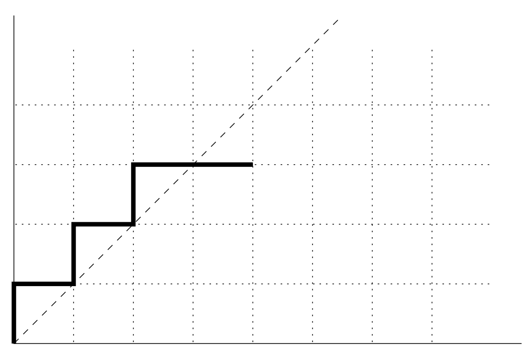
$\varepsilon = \eta 0 a \beta$



$\eta a \beta$



$\eta (a\text{-}1) \beta$



Example 20

$$
\begin{aligned}
G_{1020} &= G_{1200} - x_2\, G_{1100} \\
&= F_{12}(x_1, x_2, x_3, x_4) - x_2 F_{11}(x_1, x_2, x_3, x_4) \\
&= x_1 {x_2}^2 + x_1 {x_3}^2 + x_1 {x_4}^2 + x_2 {x_3}^2 + x_2 {x_4}^2 \\
&\quad + x_3 {x_4}^2 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 \\
&\quad + x_2 x_3 x_4 - x_2 \big(x_1 x_2 + x_1 x_3 + x_1 x_4 \\
&\quad + x_2 x_3 + x_2 x_4 + x_3 x_4 \big) \\
&= x_1 {x_3}^2 + x_1 x_3 x_4 + x_1 {x_4}^2 - {x_2}^2 x_3 - {x_2}^2 x_4 \\
&\quad + x_2 {x_3}^2 + x_2 {x_4}^2 + x_3 {x_4}^2
\end{aligned}
$$

With the two properties:

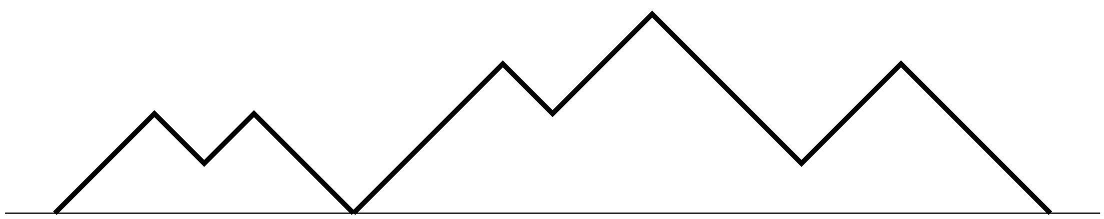
- $LM(G_\epsilon) = X_n^\epsilon$
- $\mathcal{G}$ is a Gröbner basis of $\mathcal{I}_n = \langle F_\alpha,\ |\alpha| > 0\rangle$

we prove the Theorem.

Hilbert series

$$\mathbf{SH}_n^{(k)} = \pi_k(\mathbf{SH}_n).$$

Let $C_n^{(k)}$ denote the number of Dyck paths of length $2n$ with exactly k steps down at the end.

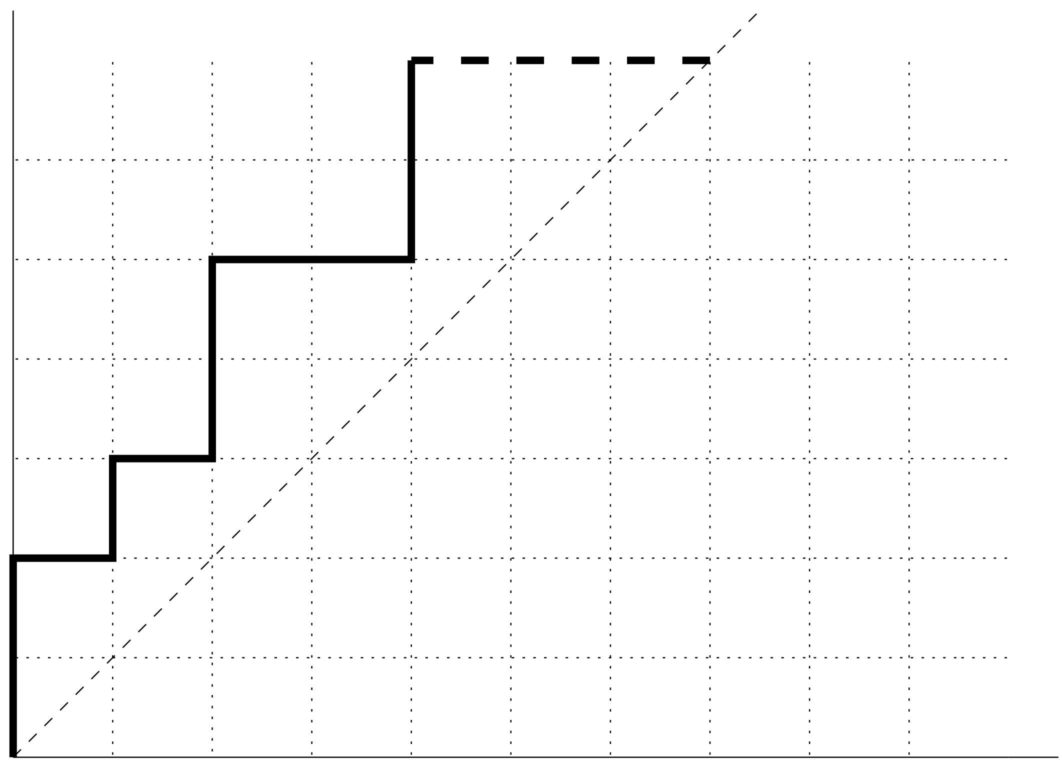


A Dyck path of length 20 with 3 final down steps.

Theorem 21

$$\dim \mathbf{SH}_n^{(k)} = C_n^{(n-k)}.$$

Example 22 For $n = 7$, the path



is associated to the monomial $x_3 x_4 x_6^2$ of degree $4 = 7 - 3$.

$$F_n(t) = \sum_{k\geq 0} \dim \mathbf{SH}_n^{(k)} t^k.$$

n	$F_n(t)$
1	1
2	$1+t$
3	$1+2t+2t^2$
4	$1+3t+5t^2+5t^3$
5	$1+4t+9t^2+14t^3+14t^4$
6	$1+5t+14t^2+28t^3+42t^4+42t^5$
7	$1+6t+20t^2+48t^3+90t^4+132t^5+132t^6$

Definition 23

$G_{n,m}$: *generalized symmetric group*

$= C_m \wr \mathcal{S}_n$

: group of matrices of pseudo-permutations whose non-zero entries are m-th roots of unity.

Example 24

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in G_{5,4}$$

Remark 25 $\sharp\, G_{n,m} = m^n\, n!$

$B_n = G_{n,2}$: *hyperoctahedral group*

: group of signed permutations.

$\mathcal{S}_n = G_{n,1}$

Action of $G_{n,m}$ on $\mathbb{C}[X_n]$:

$$g.P(X_n) = P(X_n.g)$$

Example 26 $m = 3,\ j = e^{\frac{i2\pi}{3}},\ n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j & 0 & 0 \end{pmatrix} . (x_1^2\, x_2) = j^2\, x_1\, x_3^2.$$

Definition 27 Invariants :

$$P \in Inv_{n,m} \ \Leftrightarrow\ \forall g \in G_{n,m},\ g\,.\,P = P.$$

Proposition 28

$$P \in Inv_{n,m} \ \Leftrightarrow\ \exists Q \in Sym_n \ /\ P(X_n) = Q(X_n^m).$$

Proof.

$\Leftarrow$: clear

$\Rightarrow$:

- $S_n \subset G_{n,m} \Rightarrow P \in Sym$
- $\mathbf{p} = (p_1, \ldots, p_n)$ with $m \not| \, p_j$

  $$g_j = \begin{bmatrix} 1 & \ldots & j-1 & j & j+1 & \ldots & n \\ 1 & \ldots & j-1 & \zeta j & j+1 & \ldots & n \end{bmatrix}$$

  $$P = \frac{1}{m}(Id + g_j + g_j^2 + \cdots + g_j^{m-1})\, P$$

  contains no $X_n^{\mathbf{p}}$.

Definition 29 Covariant polynomials:

$$Cov_{n,m} = \{P \in \mathbb{C}[X_n] \ / \ \forall k > 0,\ e_k(\partial X_n^m)\, P = 0\}$$
$$\simeq \mathbb{C}[X_n]/\langle e_k(X_n^m),\ \ k > 0\rangle.$$

Remark 30 $Cov_{n,1} = \mathbf{H}_n$.

Theorem 31 (Chevalley)

$$\dim Cov_{n,m} = \sharp\, G_{n,m} = m^n\, n!\,.$$

Lemma 32 *Let I be an ideal of $\mathbb{C}[X_n]$*

$$I^m = \langle P(X_n^m)\ ,\quad P \in I \rangle.$$

$\mathcal{G} = \{g_1(X_n), \ldots, g_k(X_n)\}$ *BdG of I*

$$\Downarrow$$

$\mathcal{G}^m = \{g_1(X_n^m), \ldots, g_k(X_n^m\}$ *BdG of I^m.*

Proof. Application of Buchberger's criterion :

If $\forall i \neq j\ ,\quad \overline{S(g_i, g_j)}^{\mathcal{G}} = 0$,

then $\overline{S(g_i(X_n^m), g_j(X_n^m))}^{\mathcal{G}^m} = 0$.

LM in $\langle e_k,\ k>0\rangle$:

$$x_1,\ x_2^2,\ x_3^3,\ldots,\ x_n^n.$$

LM in $\langle e_k(X^m),\ k>0\rangle$:

$$x_1^m,\ x_2^{2m},\ x_3^{3m},\ldots,\ x_n^{nm}.$$

We get a basis of the quotient:

$$\{X_n^\epsilon\ ,\quad 0\le \epsilon_i < im\}.$$

Corollary 33

$$\begin{aligned} F_q(Cov_{n,m}) &= (1+q+\cdots+q^{m-1})F_{q^m}(\mathbf{H}_n) \\ &= [m-1]_q[n!]_{q^m} \end{aligned}$$

Quasi-symmetrizing action of $G_{n,m}$ on $\mathbb{C}[X_n]$:

$$g \bullet A^K = w(g)^{c(K)} (A.|g|)_<{}^K$$

Example 34 $m = 3$, $j = e^{\frac{2i\pi}{3}}$, $n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j & 0 & 0 \end{pmatrix} \bullet (x_1^2 x_2)$$

$$\begin{aligned}
&= (j^2)^1 \left[(x_1, x_2) \, . \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right]_<^{(2,1)} \\
&= j^2 (x_3, x_1)_<{}^{(2,1)} \\
&= j^2 (x_1, x_3)^{(2,1)} \\
&= j^2 \, x_1^2 \, x_3.
\end{aligned}$$

Definition 35 Quasi-invariants:

$$P \in QInv_{n,m} \;\Leftrightarrow\; \forall g \in G_{n,m},\; g \bullet P = P.$$

Proposition 36

$$P \in QInv_{n,m} \;\Leftrightarrow\; \exists Q \in QSym_n \;/\; P(X_n) = Q(X_n^m).$$

Example 37

$$\begin{aligned} M_{2,1}(X_3) &= x_1^2\, x_2 + x_1^2\, x_3 + x_2^2\, x_3 \\ &\in QSym_3 \end{aligned}$$

$$\begin{aligned} M_{2,1}(X_3^2) &= x_1^4\, x_2^2 + x_1^4\, x_3^2 + x_2^4\, x_3^2 \\ &\in QInv_{3,2} \end{aligned}$$

Definition 38 Super-covariants:

$$SCov_{n,m} = \{P \in \mathbb{C}[X_n] \,/\, \forall |\alpha| > 0,\ M_\alpha(\partial X_n^m) P = 0\}$$
$$\simeq \mathbb{C}[X_n] / \langle M_\alpha(X_n^m),\ |\alpha| > 0 \rangle.$$

Theorem 39

$$\dim SCov_{n,m} = m^n\, C_n.$$

Theorem 40 *The set of monomials*

$$\{(X_n)^{m\,\eta+\alpha} /\ \eta\ \ Dyck,\ 0 \le \alpha_i < m\}$$

is a basis of $\mathbb{C}[X_n] / \langle M_\alpha(X_n^m),\ |\alpha| > 0 \rangle$.

LM in $\langle M_\alpha, \ |\alpha| > 0\rangle$:

$$X_n^\epsilon \quad , \quad \epsilon \text{ transdiagonal}$$

LM in $\langle M_\alpha(X_n^m), \ |\alpha| > 0\rangle$:

$$X_n^{m\epsilon} \quad , \quad \epsilon \text{ transdiagonal}$$

We get a basis of the quotient.

Group	$\mathcal{S}_n$	$G_{n,m}$
Invariants Covariants	Sym_n $\dim \mathbf{H}_n = n!$	$Inv_{n,m}$ $\dim Cov_{n,m} = m^n\, n!$
Quasi-invariants Super-covariants	$QSym_n$ $\dim \mathbf{SH}_n = C_n$	$QInv_{n,m}$ $\dim SCov_{n,m} = m^n\, C_n$