Quasi-invariant and super-coinvariant polynomials for the generalized symmetric group

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Gröbner bases.

$X_n = x_1, x_2, \ldots, x_n$

Graded lexicographic order in $\mathbb{C}[X_n]$

Example:

$$P = 2x_1^3\, x_2 - x_1\, x_2\, x_3^2 + x_2^4 - 3x_2\, x_3^2\ , \quad LM(P) = x_1^3\, x_2$$

Definition: A set $\mathcal{G} = \{g_1, \ldots, g_m\} \subset I$ is a Gröbner basis if:

$$\forall P \in I\ , \quad \exists g_i \in \mathcal{G}\ /\ LM(g_i)\,|\,LM(P).$$

Buchberger's criterion

A set $\mathcal{G} = \{g_1, \ldots, g_m\}$ is a Gröbner basis if and only if

$$\forall i \neq j, \quad \overline{S(g_i, g_j)}^{\mathcal{G}} = 0.$$

Lemma

Let $\{g_1, \ldots, g_m\}$ be a Gröbner basis of I. Then

$$\mathcal{B} = \{X_n^{\mathbf{p}} \ / \ \forall i, \ LM(g_i) \not| \ X_n^{\mathbf{p}}\}$$

is a basis of the quotient $\mathbb{C}[X_n]/I$.

Syzygies

$F = \{f_1, f_2\}$ with

$f_1 = x_1^3 + 2x_1x_2^2 + x_2^3 \quad \text{and} \quad f_2 = x_1x_2 - x_2^2$

$$\begin{aligned}
S(f_1, f_2) &= x_2\, f_1 - x_1^2\, f_2 \\
&= x_2(x_1^3 + 2x_1x_2^2 + x_2^3) \\
&\quad -x_1^2(x_1x_2 - x_2^2) \\
&= 2x_1x_2^3 + x_2^4 - x_1^2x_2^2 \\
&= -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\
&\equiv -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\
&\quad +x_1x_2(x_1x_2 - x_2^2) \\
&= x_1x_2^3 + x_2^4 \\
&\equiv x_1x_2^3 + x_2^4 - x_2^2(x_1x_2 - x_2^2) \\
&= 2x_2^4 \\
&= \overline{S(f_1, f_2)}^F
\end{aligned}$$

Quasi-symmetric polynomials

$\alpha = (\alpha_1, \ldots, \alpha_l)$ a composition, we define

$$M_\alpha(X_n) = \sum_{0 \le i_1 < \cdots < i_l \le n} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

$$F_\alpha(X_n) = \sum_{\beta \succeq \alpha} M_\beta(X_n).$$

Example: $n = 3, \alpha = (1, 2)$:

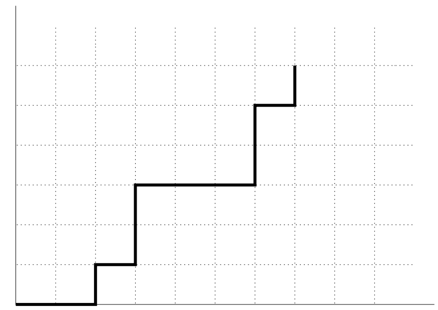
$$\begin{aligned} M_\alpha &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \\ F_\alpha &= M_{(1,2)} + M_{(1,1,1)} \\ &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3 \end{aligned}$$

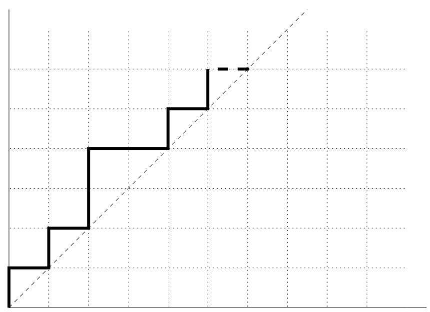
$$F_{(3,2)} = M_{(3,2)} + M_{(1,2,2)} + M_{(2,1,2)} + M_{(1,1,1,2)}$$

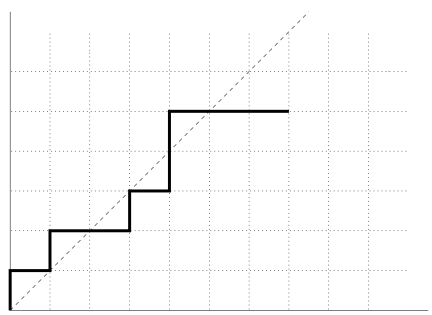
$$+ M_{(3,1,1)} + M_{(1,2,1,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}.$$

A Gröbner basis for $QSym$

Coding vectors in $\mathbb{N}^n$ by plane paths

$x_1^2 x_2 x_4^3 x_6 \longleftrightarrow \epsilon = (2, 1, 0, 3, 0, 1) \longleftrightarrow$ 

Dyck path: 

Transdiagonal path: 

Theorem [Aval, Bergeron, Bergeron]:
The leading monomials of a Gröbner basis of $\langle QSym_n \rangle$ are:

$$\{X_n^\epsilon \ / \ \epsilon \ \textit{transdiagonal}\}.$$

Definitions

$G_{n,m}$: *generalized symmetric group*

$= C_m \wr \mathcal{S}_n$

: group of matrices of pseudo-permutations whose non-zero entries are m-th roots of unity.

Let $\zeta = e^{\frac{2i\pi}{m}}$.

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in G_{5,4}$$

$\sharp\, G_{n,m} = m^n\, n!$

$B_n = G_{n,2}$: *hyperoctahedral group*

: group of signed permutations.

$\mathcal{S}_n = G_{n,1}$: *symmetric group*

Classical action of $G_{n,m}$ on $\mathbb{C}[X_n]$

$$g.P(X_n) = P(X_n.g)$$

Example: $m = 3,\ n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta \\ \zeta & 0 & 0 \end{pmatrix} . \, (x_1^2 \, x_2) = \zeta^2 \, x_1 \, x_3^2.$$

Invariants

$$P \in Inv_{n,m} \;\Leftrightarrow\; \forall g \in G_{n,m},\; g\,.\,P = P.$$

Remark : $Inv_{n,1} = Sym_n$: symmetric functions.

Characterization:

$$P \in Inv_{n,m} \;\Leftrightarrow\; \exists Q \in Sym_n \;/\; P(X_n) = Q(X_n^m).$$

Proof.

$\Leftarrow$: clear

$\Rightarrow$:

- $S_n \subset G_{n,m} \Rightarrow P \in Sym_n$
- $\mathbf{p} = (p_1, \ldots, p_n)$ with $m \not| \, p_j$

  $$g_j = \begin{bmatrix} 1 & \ldots & j-1 & j & j+1 & \ldots & n \\ 1 & \ldots & j-1 & \zeta j & j+1 & \ldots & n \end{bmatrix}$$

  $$P = \frac{1}{m}(Id + g_j + g_j^2 + \cdots + g_j^{m-1})\,P$$

  contains no $X_n^{\mathbf{p}}$.

A Gröbner basis for Sym_n $(m = 1)$

$$
\begin{aligned}
h_1(X_4) &= x_1 + x_2 + x_3 + x_4 \\
h_2(X_4) &= x_1\, h_1(X_4) + h_2(x_2, x_3, x_4) \\
&\equiv h_2(x_2, x_3, x_4) \\
h_3(X_4) &= x_1\, h_2(X_4) + h_3(x_2, x_3, x_4) \\
&\equiv h_3(x_2, x_3, x_4) \\
&= x_2\, h_2(x_2, x_3, x_4) + h_3(x_3, x_4) \\
&\equiv h_3(x_3, x_4) \\
h_4(X_4) &= \cdots \equiv h_4(x_4)
\end{aligned}
$$

Leading Monomials: x_1, x_2^2, x_3^3, x_4^4.

In general: $x_1, x_2^2, \ldots, x_n^n$.

This is a Gröbner basis of $\langle Sym_n \rangle$.

Useful lemma

Let I be an ideal of $\mathbb{C}[X_n]$

$$I^m = \langle P(X_n^m)\ ,\quad P \in I\rangle.$$

$\mathcal{G} = \{g_1(X_n), \ldots, g_k(X_n)\}$ a Gröbner basis of I

$$\Downarrow$$

$\mathcal{G}^m = \{g_1(X_n^m), \ldots, g_k(X_n^m\}$ a Gröbner basis of I^m.

Proof.

Application of Buchberger's criterion :

If $\forall i \neq j$, $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$,

then $\overline{S(g_i(X_n^m), g_j(X_n^m))}^{\mathcal{G}^m} = 0$.

Example:

$g_1 = x_1x_2 + x_1x_3 + x_2x_3,\ g_2 = x_2^2 + x_3^2,$
$g_3 = x_2x_3^2 + x_3^3$

$$
\begin{aligned}
S(g_1, g_2) &= x_2g_1 - x_1g_2 = x_1x_2x_3 + x_2^2x_3 - x_1x_3^2 \\
&\equiv x_1x_2x_3 + x_2^2x_3 - x_1x_3^2 - x_3g_1 \\
&\equiv x_2^2x_3 - x_3^2x_2 \\
&\equiv x_2^2x_3 - x_3^2x_2 - x_3g_2 \\
&\equiv -x_2x_3^2 - x_3^3 = -g_3 \equiv 0
\end{aligned}
$$

A Gröbner basis for $Inv_{n,m}$

A Gröbner basis when $m = 1$ is

$$\{h_k(x_k, \dots, x_n)\}$$

and the Leading Monomials are :

$$x_1,\ x_2^2,\ x_3^3, \dots,\ x_n^n.$$

A Gröbner basis for $m \in \mathbb{N}$ is

$$\{h_k(x_k^m, \dots, x_n^m)\}$$

and the Leading Monomials are

$$x_1^m,\ x_2^{2m},\ x_3^{3m}, \dots,\ x_n^{nm}.$$

Coinvariant polynomials

$$Q_{n,m} = \mathbb{C}[X_n]/\langle Inv_{n,m} \rangle$$

We get a basis of this quotient:

$$\{X_n^\epsilon \; / \;\; 0 \leq \epsilon_i < im\}.$$

Theorem [Chevalley]:

$$\dim Q_{n,m} = \sharp\, G_{n,m} = m^n \, n!\,.$$

Corollary: Hilbert series

$$F_q(Q_{n,m}) \quad = \quad [m]_q^n \, [n!]_{q^m}$$

Quasi-symmetrizing action of $G_{n,m}$

$$g \bullet A^K = w(g)^{c(K)} (A.|g|)_<{}^K$$

Example: $m = 3$, $\zeta = e^{\frac{2i\pi}{3}}$, $n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta \\ \zeta & 0 & 0 \end{pmatrix} \bullet (x_1^2 x_2)$$

$$\begin{aligned}
&= (\zeta^2)^1 \left[(x_1, x_2) \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right]_<^{(2,1)} \\
&= \zeta^2 (x_3, x_1)_<{}^{(2,1)} \\
&= \zeta^2 (x_1, x_3)^{(2,1)} \\
&= \zeta^2 x_1^2 x_3.
\end{aligned}$$

Quasi-invariants

$$P \in QInv_{n,m} \;\Leftrightarrow\; \forall g \in G_{n,m},\; g \bullet P = P.$$

Remark: $QInv_{n,1} = QSym_n$ [Hivert]
(Quasi-symmetric polynomials)

Characterization

$$P \in QInv_{n,m} \;\Leftrightarrow\; \exists Q \in QSym_n \;/\; P(X_n) = Q(X_n^m).$$

Super-coinvariant polynomials

$$SQ_{n,m} = \mathbb{C}[X_n]/\langle QInv_{n,m}\rangle.$$

Theorem:

$$\dim SQ_{n,m} = m^n\, C_n.$$

Theorem: The set of monomials

$$\{(X_n)^{m\,\eta+\alpha} /\ \eta\ \ Dyck,\ 0 \leq \alpha_i < m\}$$

is a basis of $SQ_{n,m}$.

A Gröbner basis for $QInv_{n,m}$

LM in $\langle M_\alpha, \ |\alpha| > 0\rangle$:

$$X_n^\epsilon \quad , \quad \epsilon \text{ transdiagonal}$$

LM in $\langle M_\alpha(X_n^m), \ |\alpha| > 0\rangle$:

$$X_n^{m\epsilon} \quad , \quad \epsilon \text{ transdiagonal}$$